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CALCULATION OF THE TRANSIENT MOTION OF ELASTIC AIRFOILS FORCED BY CONTROL SURFACE MOTION AND GUSTS

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INTRODUCTION

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This report continues the development of the equations of motion of a three-degree-of-freedom typical section in two-dimensional, incompressible flow begun by Balakrishnan in reference 1. The goal of this research is to investigate active aeroelastic control concepts such as flutter control and gust alleviation, although the present study is of independent interest as a development of unsteady aerodynamic theory.

The design of active aeroelastic control schemes is greatly aided by the availability of mathematical models valid for arbitrary motions. Reference 2 discusses methods of deriving such models from airloads derived for oscillatory motions; references 3 and 4 present similar formulations for Theodorsen's problem involving inverse Fourier transformation to obtain impulse response function airloads for use in convolution integral solutions of the equations of motion. Although these formulations are mathematically correct, the calculations are cumbersome and involve functions available only in tabulated form. Consequently, few examples of the exact transient response of airfoils excited by either control surface motion or gusts have been calculated. More common are calculations using finite state, rational function approximations for the unsteady aerodynamic airloads.

In order to provide a basis for the analysis of aeroelastic control schemes, this report develops the exact transient response of a three-degree-of-freedom airfoil (pitch, plunge, and flap) forces by flap motions and gusts. The development makes extensive use of special time-domain functions derived from a function studied by Kussner (Sears, ref. 5).

Chapter 2 summarizes the basic equations of incompressible, two-dimensional flow and Chapter 3 derives expressions for the circulation on the airfoil. Chapters 4, 5, and 6 derive the lift, pitching moment, and flap moment expressions, while Chapter 7 adds the section dynamics to give the complete equations of motion as a set of coupled, integro-differential equations. Chapter 8 then gives the steady state airfoil displacements, pressure distribution, and airloads due to flap deflection. In Chapter 9, approximations to the transient response model are introduced for the special cases of zero stream velocity, small time, and large time. In addition, a numerical solution technique is given for the solution of the general case and examples of the exact transient response of an airfoil at several speeds are presented. Chapter 10 completes the report with the development of the airloads upon the airfoil due to the penetration of a frozen gust field represented as a stationary Gaussian random process. Two appendices summarize the integral formulas and special functions used in the report.

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THE AERODYNAMIC MODEL: BASIC EQUATIONS

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2. The Aerodynamic Model.

We consider a typical section, of mass per unit length m_s , extending along the X-axis from -1 to +1 , with motion entirely in the X-Z plane. Let $\phi(t,x,z)$ denote the velocity potential. Then

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad t > 0, \quad x < 1$$

with the boundary conditions:

1. Flow Tangency Condition:

$$\frac{\partial \phi}{\partial z}(t,x,0+) = w_a(t,x) \quad -1 < x < +1; \quad 0 < t$$

where $w_a(t,x)$ is the downwash to be specified later.

2. Zero Pressure-Discontinuity Condition:

$$\begin{aligned} \psi(t,x,0+) - \psi(t,x,0-) &= 0 \qquad 1 \le x \le 1 + Ut \\ &= 0, \qquad x = 1-, \qquad \text{(Kutta Condition)} \end{aligned}$$

where $\psi(t,x,z)$ is the acceleration potential defined by

$$\psi(t,x,z) = U \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}$$

The perturbation pressure is given by $P = -\rho\psi$.

Following Schwarz (see Ashley [3]), we seek a solution in the form:

$$\phi(t,x,z) = -(1/2\pi) \int_{-1}^{1} \gamma a(t,\zeta) \; \mathrm{Tan}^{-1} \frac{z}{x - \zeta} \, d\zeta - (1/2\pi) \int_{1}^{1+Ut} \gamma_w(t,\zeta) \; \mathrm{Tan}^{-1} \frac{z}{x - \zeta} \, d\zeta$$

where $\gamma_a(t,x)$ is the circulation on the foil, and $\gamma_w(t,x)$ the circulation in the wake. The integrals are to be taken in the Cauchy sense.

It is easily shown that:

$$\frac{\partial\phi}{\partial x}(t,x,0+) = \tfrac{1}{2}\gamma_a(t,x) \qquad -1 < x < +1$$

$$= \tfrac{1}{2}\gamma_w(t,x) \qquad +1 < x < 1 + Ut$$

and that

$$\frac{\partial\phi}{\partial x}(t,x,0-) = (-1) \frac{\partial\phi}{\partial x}(t,x,0+)$$

$$\frac{\partial\phi}{\partial x}(t,x,0+) = 0, \qquad x < -1, \quad x > 1 + Ut$$

Hence, in particular:

$$\phi(t,x,0+) = \tfrac{1}{2} \int_{-1}^{x} \gamma_a(t,\zeta)\, d\zeta \qquad -1 < x < +1$$

$$= \tfrac{1}{2} \left\{ \int_{-1}^{1} \gamma_a(t,\zeta)\, d\zeta + \int_{-1}^{x} \gamma_w(t,\zeta)\, d\zeta \right\}, \qquad 1 < x < 1 + Ut$$

Again, even though we cannot differentiate with respect to time inside the integrals, we have

$$-\frac{\partial}{\partial t}\phi(t,x,|z|) = \frac{\partial}{\partial t}\phi(t,x,-|z|)$$

by considering corresponding differentials. In particular

$$-\frac{\partial}{\partial t}\phi(t,x,0+) = \frac{\partial}{\partial t}\phi(t,x,0-)$$

Hence the Boundary Condition 2 yields:

$$U\gamma_w(t,1+) + \Gamma'(t) = 0$$

And Kutta Condition yields:

$$U\gamma_a(t,1-) + \Gamma'(t) = 0$$

Hence,

$$\frac{-\Gamma'(t)}{U} = \gamma_a(t,1-) = \gamma_w(t,1+) \quad \text{and is finite.}$$

Moreover, we have, from boundary condition 2:

$$U\gamma_w(t,x) + \Gamma'(t) + \frac{\partial}{\partial t}\int_1^x \gamma_w(t,y)\,dy = 0 \qquad 1 \le x \le 1 + Ut$$

Hence $\gamma_w(t,x)$ must satisfy:

$$\gamma_w(t,x) = F(t - \frac{x}{U})$$

Putting $x = 1+$, we have

$$\gamma_w(t,1+) = F(t - \frac{1}{U}) = \frac{-\Gamma'(t)}{U}$$

or,

$$\Gamma'(t) = U \; F(t - 1/U)$$

or,

$$F(t) = \frac{\Gamma'(t + 1/U)}{U} \quad , \quad t \geq 0$$

Hence

$$\gamma_w(t,x) = -\frac{1}{U}\Gamma'(t + \frac{1-x}{U})$$

Now

$$w_a(t,x) = \frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma_a(t,y)dy}{x-y} - \frac{1}{2\pi}\int_{1}^{1+Ut}\frac{\gamma_w(t,y)dy}{x-y}$$

Substituting for $\gamma_w(t,x)$, we get:

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma_a(t,y)dy}{x-y} = w_a(t,x) - \frac{1}{2\pi}\int_{1}^{1+Ut}\frac{\Gamma'(t+\frac{1-y}{U})dy}{x-y}$$

$$= w_a(t,x) - \frac{1}{2\pi}\int_{0}^{t}\frac{\Gamma'(t-\sigma)d\sigma}{x-1-U\sigma} \quad , \quad -1 < x < 1$$

Now, this integral equation has a unique solution, since $\gamma_a(t,1-)$ is required to be finite; and is given by:

$$\gamma_a(t,x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{\int_{-1}^{1}\frac{\sqrt{1+y}}{\sqrt{1-y}}\frac{w_a(t,y)}{x-y}dy - \frac{1}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\frac{dy}{x-y}\int_{0}^{t}\frac{\Gamma'(t-\sigma)d\sigma}{y-1-U\sigma}\right\}$$

This is our basic integral equation.

Let

$$H(\sigma,x) = \int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\frac{1}{y-1-U\sigma}\cdot\frac{1}{x-y}dy = (-\pi)\frac{1}{(x-z)}\sqrt{\frac{z+1}{z-1}}; \qquad z = 1+U\sigma$$

Then the equation becomes;

$$\gamma_a(t,x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\,\frac{w_a(t,y)}{x-y}\,dy - \frac{1}{2\pi}\int_0^t H(\sigma,x)\,\Gamma'(t-\sigma)d\sigma\right\}$$

giving the total circulation on the airfoil as

$$\Gamma(t) = 2\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\,w_a(t,y)dy - \frac{1}{2\pi}\int_0^t H(\sigma)\,\Gamma'(t-\sigma)d\sigma$$

with

$$H(\sigma) = \frac{2}{\pi}\int_0^1\sqrt{\frac{1-x}{1+x}}\,H(\sigma,x)dx = 2\pi\left[\sqrt{\frac{z+1}{z-1}} - 1\right] \;;\quad z = 1 + U\sigma$$

(see Appendix 1.)

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CIRCULATION ON THE FOIL

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3. The Circulation on the Foil.

To proceed further, we endow the foil with three degrees of freedom. With reference to Figure 1, let $h(t)$ denote the plunge coordinate, $\alpha(t)$ the angle of attack, and $\beta(t)$ the flap deflection. Then the downwash $w_a(t,x)$ is given by

$$w_a(t,x) = \frac{\partial}{\partial t} z_a(t,x) + U \frac{\partial}{\partial x} z_a(t,x)$$

where

$$z_a(t,x) = -h(t) - (x - a)\,\alpha(t), \qquad -1 < x < c$$

$$= -h(t) - (x - a)\,\alpha(t) - (x - c)\,\beta(t), \qquad c < x < +1$$

Then

$$w_a(t,x) = -h'(t) - (x-a)\alpha'(t) - U\alpha(t), \quad -1 < x < c$$

$$= -h'(t) - (x-a)\alpha'(t) - U\alpha(t) - (x-c)\beta'(t) - U\beta(t), \quad c < x < 1$$

Hence we have for the circulation:

$$\Gamma(t) = \int_{-1}^{1} \gamma_a(t,x)dx$$

$$= \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{dx}{x-\zeta} \left\{ \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} w_a(t,\zeta)d\zeta - \frac{1}{2\pi} \int_{0}^{t} H(\sigma)\, \Gamma'(t-\sigma)d\sigma \right\}$$

$$= (-2) \int_{-1}^{1} \frac{\sqrt{1+\sigma}}{\sqrt{1-\sigma}} w_a(t,\sigma)d\sigma - \frac{1}{2\pi} \int_{0}^{t} H(\sigma)\, \Gamma'(t-\sigma)d\sigma$$

The last line follows since

$$\frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{dx}{x-\zeta} = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(\frac{1-\zeta}{x-\zeta} - 1\right)dx = -2.$$

Also

$$\frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{dx}{x-\zeta} \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} w_a(t,\zeta)d\zeta = -2 \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} w_a(t,\zeta)d\zeta$$

$$= 2 \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (h'(t) + (\zeta-a)\alpha'(t) + U\alpha(t))d\zeta + 2 \int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} ((\zeta-c)\beta'(t) + U\beta(t))d\zeta$$

$$= 2\pi h'(t) + 2\pi U\alpha(t) + (\pi-2\pi a)\alpha'(t) + (2\sqrt{1-c^2} + (1-2c)\cos^{-1}c - c\sqrt{1-c^2})\beta'(t)$$

$$+ (2\cos^{-1}c + 2\sqrt{1-c^2})U\beta(t).$$

Hence

$$\Gamma(t) = 2\pi U\alpha(t) + (2\cos^{-1}c + 2\sqrt{1-c^2})U\beta(t)$$

$$+ 2\pi h'(t) + (\pi-2\pi a)\alpha'(t) + (2\sqrt{1-c^2} + (1-2c)\cos^{-1}c - c\sqrt{1-c^2})\beta'(t)$$

$$+ \int_0^t (1 - \frac{\sqrt{z+1}}{\sqrt{z-1}})\Gamma'(t-\sigma)d\sigma.$$

Let $$x(t) = \begin{vmatrix} h(t) \\ \alpha(t) \\ \beta(t) \end{vmatrix}$$

Hence

$$0 = [B, Z(t) - Z(0)] - \int_0^t \frac{\sqrt{z+1}}{\sqrt{z-1}} \Gamma'(t-\sigma)\,d\sigma$$

Now for Res > 0:

$$\int_0^\infty e^{-s\sigma} \frac{\sqrt{z+1}}{\sqrt{z-1}}\, d\sigma = \frac{1}{U} \int_1^\infty e^{-\frac{s}{U}(z-1)} \frac{\sqrt{z+1}}{\sqrt{z-1}}\, dz$$

$$= \frac{1}{U} e^{+\frac{s}{U}} \int_1^\infty e^{-\frac{sz}{U}} \frac{\sqrt{z+1}}{\sqrt{z-1}}\, dz$$

Now

$$\int_1^\infty e^{-\frac{sz}{U}} \frac{\sqrt{z+1}}{\sqrt{z-1}}\, dz = \int_1^\infty e^{-\frac{sz}{U}} \frac{(z+1)}{\sqrt{z^2-1}}\, dz$$

Let

$$K_0(s) = \int_1^\infty e^{-st} \frac{dt}{\sqrt{t^2-1}}$$

Then

$$\int_1^\infty e^{-st} \frac{t\,dt}{\sqrt{t^2-1}} = -K_0'(s)$$

Hence

$$\int_1^\infty e^{-\frac{sz}{U}} \frac{\sqrt{z+1}}{\sqrt{z-1}}\,dz = K_0(\tfrac{s}{U}) - K_0'(\tfrac{s}{U})$$

Here $K_0(s)$ is the modified Bessel function of order zero.

Hence

$$\int_0^\infty e^{-s\sigma} \frac{\sqrt{z+1}}{\sqrt{z-1}}\,d\sigma = \frac{1}{U} e^{s/U} (K_0(\tfrac{s}{U}) - K_0'(s/U))$$

and hence

$$\int_0^\infty e^{-s\sigma}\Gamma'(\sigma)d\sigma = \frac{U[B, L(Z;s) - \frac{Z(0)}{s}]e^{-s/U}}{(K_0(\frac{s}{U}) - K_o'(s/U))}$$

where

$$L(Z;s) = \int_0^\infty e^{-st} Z(t)\,dt$$

Let us denote the inverse Laplace transform of

$$\frac{1}{s}\,\frac{U\,e^{-s/U}}{K_0(\frac{s}{U}) - K_o'(s/U)} = \frac{1}{s}\,\frac{1}{\int_0^\infty e^{-s\sigma}\,\frac{\sqrt{z+1}}{\sqrt{z-1}}\,d\sigma}$$

by

$$c_1(t).$$

Then

$$\Gamma'(t) = \int_0^t c_1(t-\sigma)[B,\dot{Z}(\sigma)]\,d\sigma$$

and

$$\Gamma(t) = \int_0^t c_1(t-\sigma)[B,Z(\sigma)]d\sigma - c_2(t)[B,Z(0)] + [B,Z(0)]$$

where

$$c_2(t) = \int_0^t c_1(s)ds$$

Finally we note the series expansion due to Kussner-Sears [5] for $c_1(t)$

$Ut \leq 5$

$$c_1(t) = U \frac{\sqrt{2}}{\pi} \{ \frac{1}{2} (Ut)^{-1/2} - \frac{1}{8} (Ut)^{1/2} + \frac{5}{192} (Ut)^{3/2} - \frac{161}{26880} (Ut)^{5/2} \ldots \}$$

$Ut \geq 5$

$$c_1(t) = \frac{U}{(Ut-1)^2} \{1 - \frac{5}{(Ut-1)} + \frac{4 \text{ Log } (2Ut - 2)}{(Ut-1)} - \frac{54 \text{ Log } (2Ut - 2)}{(Ut-1)^2} + \frac{(14 + \frac{9}{2} - 3\pi^2)}{(Ut-1)^2}$$
$$+ \frac{18 (\text{Log } (2Ut - 2))^2}{(Ut-1)^2} \ldots \}$$

$$c_1(t;U) = U\, c_1(Ut;1)$$

$$c_2(t;U) = c_2(Ut;1)$$

Figures 2 and 3 present plots of $c_1(t)$ and $c_2(t)$ calculated using the above approximations.

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CALCULATION OF THE LIFT

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4. Calculation of the Lift P.

We have

$$P = \int_{-1}^{1} P(x,t)dx$$

$$= (-\rho) \left[\int_{-1}^{1} U\gamma_a(t,x)dx + \frac{\partial}{\partial t} \int_{-1}^{1} \int_{-1}^{x} \gamma_a(t,y)dy\, dx \right]$$

$$= (-\rho) \left[U\,\Gamma(t) - \frac{\partial}{\partial t} \int_{-1}^{1} x\gamma_a(t,x)dx + \Gamma'(t) \right]$$

Now

$$\frac{\partial}{\partial t} \int_{-1}^{1} x\gamma_a(t,x)dx = \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}}\, x\, dx \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\frac{\partial}{\partial t} w_a(t;\zeta)}{x-\zeta}\, d\zeta$$

$$- \frac{\partial}{\partial t}\left[\int_{0}^{t}\int_{-1}^{1} \frac{2}{\pi} \sqrt{\frac{1-x}{1+x}}\; \frac{x}{2\pi} H(\sigma,x)dx\, \Gamma'(t-\sigma)d\sigma \right] \equiv T_1 + T_2$$

Now

$$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\frac{\partial}{\partial t} w_a(t,\zeta)}{x-\zeta}\, d\zeta = \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} (-h''(t) - (\zeta-a)\alpha''(t) - U\,\alpha'(t))d\zeta$$

$$+ \int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} (-(\zeta-c)\beta''(t) - U\,\beta'(t))d\zeta$$

$$= +\pi\, h''(t) - (a\pi - \pi \;\; - \pi x)\, \alpha''(t) + U\pi\, \alpha'(t) + (\cos^{-1} c + \sqrt{1-c^2})\, \beta''(t)$$

$$+ (x-c)\,(\cos^{-1} c + \sqrt{\frac{1+x}{1-x}} \quad \mathrm{Log}\, |\,|)\, \beta''(t)$$

$$+ U\, \beta'(t)\, (\cos^{-1} c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \quad \mathrm{Log}\, |\,|\,)$$

where

$$\mathrm{Log}\, |\cdot| \;\; = \log \frac{\sqrt{(1-c)(1+x)} + \sqrt{(1+c)(1-x)}}{\left|\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}\,\right|}$$

$$\frac{d}{dx} \;\; \mathrm{Log}\;\; |\cdot| \;\; = \; - \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \;\cdot\; \frac{1}{x-c}$$

and where we note that

$$\int_{-1}^{1} x(1+x)\, \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx = 0; \quad \int_{-1}^{1} x \log |\,|\; dx = \frac{\pi}{2}\, c\, \sqrt{1-c^2}; \quad \int_{-1}^{1} x(x-c) \log |\cdot|\; dx = \frac{\pi}{6}\, (1-c^2)^{3/2}$$

$$\int_{-1}^{1} x(x-c) \;\; \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx = \frac{\pi}{2}\, (1+c); \quad \int_{-1}^{1} x \;\; \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx = \; - \frac{\pi}{2}$$

Hence

$$T_1 = -\pi h''(t) + (a\pi)\,\alpha''(t) - U\,\pi\alpha'(t) + \left(c\cos^{-1}c - \sqrt{1-c^2} + \frac{1}{3}\,(\sqrt{1-c^2})^3\right)\beta''(t) + U(c\sqrt{1-c^2} - \cos^{-1}c)\beta'(t)$$

Next

$$T_2 = \frac{\partial}{\partial t}\left[\int_0^t \int_{-1}^{1} \frac{2}{\pi}\,\frac{\sqrt{1-x}}{\sqrt{1+x}}\cdot x\cdot\left(-\frac{1}{2\pi}\right) H(\sigma,x)dx\,\Gamma'(t-\sigma)d\sigma\right]$$

$$= \frac{\partial}{\partial t}\left[\int_0^t (1 + U\sigma - \sqrt{(1+U\sigma)^2 - 1}\,)\Gamma'(t-\sigma)d\sigma\right]$$

$$= +\ \Gamma'(t) + \frac{\partial}{\partial t}\int_0^t (U\sigma - \sqrt{U^2\sigma^2 + 2U\sigma})\,\Gamma'(t-\sigma)d\sigma$$

Hence

$$P = (-\rho)\ \{U\Gamma(t) + U\pi\alpha'(t) - U(c\sqrt{1-c^2} - \cos^{-1}c)\beta'(t) + \pi h''(t) - a\pi\alpha''(t)$$

$$-\left(\frac{1}{3}\,(\sqrt{1-c^2})^3 + c\cos^{-1}c - \sqrt{1-c^2}\right)\beta''(t) - \frac{d}{dt}\int_0^t c_3(t-\sigma)\,[B,\dot{Z}(\sigma)]d\sigma\}$$

where

$$c_3(t) = \int_0^t c_1(t-\sigma)\,[U\sigma - \sqrt{U^2\sigma^2 + 2U\sigma}\,]d\sigma$$

The function $c_3(t)$ may be calculated numerically using the approximations given in Chapter 3. Figure 4 presents a plot of $c_3(t)$.

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CALCULATION OF THE PITCHING MOMENT M_α

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5. Calculation of the Pitching Moment M_α.

We have:

$$M_\alpha = (-\rho) \left\{ \int_{-1}^{1} (x-a)U \, \gamma_a(t,x)dx + \frac{\partial}{\partial t} \int_{-1}^{1} (x-a)dx \int_{-1}^{x} \gamma_a(t,y)dy \right\}$$

$$= (-\rho) \left\{ \int_{-1}^{1} U(x-a)\gamma_a(t,x)dx - \frac{\partial}{\partial t} \int_{-1}^{1} \frac{(x-a)^2}{2} \gamma_a(t,x)dx + \frac{(1-a)^2}{2} \Gamma'(t) \right\}$$

$$\int_{-1}^{1} (x-a)\dot{\gamma}_a(t,x)dx = - a\,\Gamma(t) + \int_{-1}^{1} x\gamma_a(t,x)dx$$

$$\int_{-1}^{1} x\,\gamma_a(t,x)dx = \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot x \cdot \left\{ \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{w_a(t,\zeta)}{x-\zeta} d\zeta - \frac{1}{2\pi} \int_0^t H(\sigma,x)\, \Gamma'(t-\sigma)d\sigma \right\} dx$$

$$= \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}} \, x \, dx \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{w_a(t,\zeta)}{x-\zeta} d\zeta + \int_0^t ((1+U\sigma) - \sqrt{U^2\sigma^2+2U\sigma})\Gamma'(t-\sigma)d\sigma$$

$$= T_1 + (\Gamma(t) - \Gamma(0)) + \int_0^t c_3(t-\sigma)\, [B,\dot{Z}(\sigma)]d\sigma$$

$$T_1 = \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}}\, x \cdot \left\{ \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} [- h'(t) - (\zeta-a)\alpha'(t) - U\alpha(t)] \cdot \frac{d\zeta}{x-\zeta} \right.$$

$$\left. + \int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} [- (\zeta-c)\beta'(t) - U\,\beta(t)] \frac{d\zeta}{x-\zeta} \right\}$$

$$= \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}}\, x \cdot \quad \{\pi(h'(t) + U\,\alpha(t)) - \pi(a-1-x)\alpha'(t) + U\,(\cos^{-1}c + \sqrt{\frac{1+x}{1-x}} \log |\,|)\beta(t)$$

$$+ [(\cos^{-1}c + \sqrt{1-c^2}) \quad + (\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log |\quad| \,)(x-c)]\beta'(t)\}\, dx$$

$$= - \pi\,(h'(t) + U\alpha(t)) + \pi a\,\alpha'(t) + U(-\cos^{-1}c + c\sqrt{1-c^2})\beta(t) + [c \cos^{-1}c - \frac{1}{3}(2 + c^2)\sqrt{1-c^2}]\beta'(t)$$

Next, the "non-circulatory" terms (that is, terms not containing $\Gamma(t)$) in:

$$\frac{\partial}{\partial t} \int_{-1}^{1} \frac{(x-a)^2}{2} \quad \gamma_a(t,x)dx$$

$$= \int_{-1}^{1} \frac{1}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-a)^2 \{\pi\,(h''(t) + U\,\alpha'(t)) - \pi(a-1-x)\alpha''(t) + U\,(\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \quad \log |\cdot|)\beta'(t)$$

$$+ [(\cos^{-1}c + \sqrt{1-c^2}) + (\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log |\cdot|)(x-c)\beta''(t)\}$$

$$= (\frac{1}{2} + a^2 + a)\pi(h''(t)+U\alpha'(t)) + \pi(\frac{1}{8} - \frac{a}{2} - a^3 - \frac{a^2}{2})\alpha''(t) + U[(\frac{1}{2} + a + a^2)\cos^{-1}c$$

$$+ (\frac{1}{6} + 2a^2 + \frac{2c^2}{3} - 2ac)\sqrt{1-c^2}]\beta'(t) + [(\cos^{-1}c + \sqrt{1-c^2})(\frac{1}{2} + a^2 + a)$$

$$+ \frac{1}{12}\sqrt{1-c^2}\,(4ac^2 - 6a^2c - c^3 - 4a - \frac{c}{2}) - \cos^{-1}c(a + \frac{a^2}{2} + \frac{3}{8} + \frac{c}{2} + a^2c + ac)]\beta''(t)$$

Hence

$$M_\alpha = (-\rho)\quad\{-aU\Gamma(t) + U\,[\Gamma(t) - \Gamma(0)] + U\int_0^t c_3(t-\sigma)\,[B,\dot{Z}(\sigma)]d\sigma$$

$$+ U[-\pi h'(t) - U\alpha(t)\pi + \pi a\alpha'(t) + U\,(-\cos^{-1}c + c\sqrt{1-c^2})\beta(t) + (c\cos^{-1}c - \frac{1}{3}(2+c^2)\sqrt{1-c^2})\beta'(t)]$$

$$+ \frac{(1-a)^2}{2}\,\Gamma'(t) - (\frac{1}{2} + a^2 + a)\pi\,(h''(t) + U\alpha'(t)) - \pi(\frac{1}{8} - \frac{a}{2} - a^3 - \frac{a^2}{2})\,\alpha''(t)$$

$$- U\,[(\frac{1}{2} + a + a^2)\cos^{-1}c + (\frac{1}{6} + 2a^2 + \frac{2c^2}{3} - 2ac)\sqrt{1-c^2}]\beta'(t)$$

$$- [(\cos^{-1}c(\frac{1}{8} + \frac{a^2}{2} - \frac{c}{2} - a^2c - ac) + \sqrt{1-c^2}\,(\frac{1}{2} + a^2 + \frac{2}{3}a + \frac{1}{3}ac^2 - \frac{1}{2}a^2c - \frac{c^3}{12} - \frac{c}{24})]\beta''(t)$$

$$+ \frac{d}{dt}\frac{1}{2}\int_0^t [\frac{1}{\pi^2}\int_{-1}^{1}(x-a)^2\,\frac{\sqrt{1-x}}{\sqrt{1+x}}\,H(\sigma,x)dx]\Gamma'(t-\sigma)d\sigma\}$$

where the factor in square brackets in the integrand in the last term is

$$\frac{1}{\pi^2} \int_{-1}^{1} (x-a)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} H(\sigma,x)dx$$

$$= (\frac{1}{2} \frac{\sqrt{z+1}}{\sqrt{z-1}} - z^2 + z\sqrt{z^2-1}) + 2a\,(z - \sqrt{z^2-1}) + a^2(\frac{\sqrt{z+1}}{\sqrt{z-1}} - 1)$$

It is also possible to split the circulatory and non-circulatory terms in yet another way. Thus by <u>not</u> splitting

$$\int_{-1}^{1} (x-a)\, \gamma_a(t,x)dx = - a\Gamma(t) + \int_{-1}^{1} x\gamma_a(t,x)dx$$

but directly calculating the left side, we have:

$$\int_{-1}^{1} (x-a)\gamma_a(t,x)dx = T_3 + T_4$$

$$T_3 = \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-a)\, \{\pi h'(t) + \pi U\alpha(t) + \pi(1-x-a)\alpha'(t) + U(\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log|\ |)\beta(t)$$

$$+ [(\cos^{-1}c + \sqrt{1-c^2}) + (\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log|\ |)\,(x-c)]\beta'(t)\}\,dx$$

$$= -(1+2a)\pi U\alpha(t) - (1+2a)\pi h'(t) + 2a^2\pi\alpha'(t) + u\,((-1-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2})\beta(t)$$

$$+ [(\cos^{-1}c + \sqrt{1-c^2})(-1-2a) + 2\cos^{-1}c\,(ac + \frac{a}{2} + \frac{c}{2} + \frac{1}{2}) + \frac{1}{3}\sqrt{1-c^2}\,(1-c^2 + 3ac)]\beta'(t)$$

$$T_4 = + 2\pi \int_0^t [z - \sqrt{z^2-1} + a\,\frac{\sqrt{z+1}}{\sqrt{z-1}} - a]\,\Gamma'(t-\sigma)d\sigma$$

This yields:

$$M_\alpha = (-\rho)\,\{U^2(-1-2a)\pi\alpha(t) + U^2[(-1-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2}]\beta(t)$$

$$- U(1+2a)\pi h'(t) + 2Ua^2\pi\alpha'(t) + U[\frac{1}{3}\sqrt{1-c^2}\,(1-c^2 + 3ac) + 2\cos^{-1}c(ac + \frac{a}{2} + \frac{c}{2} + \frac{1}{2}) - (1+2a)(\cos^{-1}c + \sqrt{1-c^2})]\beta'(t)$$

$$- (\frac{1}{2} + a^2 + a)\pi\,U\,\alpha'(t) - U\,[(\frac{1}{2} + a + a^2)\cos^{-1}c + (\frac{1}{6} + 2a^2 + \frac{2c^3}{3} - 2ac)\sqrt{1-c^2}\,]\beta'(t)$$

$$- (\frac{1}{2} + a^2 + a)\pi\,h''(t) - \pi(\frac{1}{8} - \frac{a}{2} - a^3 - \frac{a^2}{2})\alpha''(t)$$

$$- [(\cos^{-1}c + \sqrt{1-c^2})(\frac{1}{2} + a^2 + a) + \frac{1}{12}\sqrt{1-c^2}\,(4ac^2 - 6a^2c - c^3 - 4a - \frac{c}{2}) - (\cos^{-1}c)(a + \frac{a^2}{2} + \frac{3}{8} + \frac{c}{2} + a^2c + ac)]\beta''(t)$$

$$+ \frac{(1-a)^2}{2}\,\Gamma'(t) + U\int_0^t (z - \sqrt{z^2-1} + a\,\frac{\sqrt{z+1}}{\sqrt{z-1}} - a)\Gamma'(t-\sigma)d\sigma$$

$$+ \frac{1}{2}\,\frac{d}{dt}\int_0^t [\frac{1}{2}\,\frac{\sqrt{z+1}}{\sqrt{z-1}} - z^2 + z\sqrt{z^2-1} + 2a\,(z - \sqrt{z^2-1}) + a^2\,(\frac{\sqrt{z+1}}{\sqrt{z-1}} - 1)]\Gamma'(t-\sigma)d\sigma$$

$$M_\alpha = (-\rho)\ \{U^2(-1-2a)\pi\ \alpha(t) + U^2[(-1-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2}]\beta(t)$$

$$-U(1+2a)\pi\ h'(t) + U\pi(a^2 - a - \frac{1}{2})\alpha'(t)$$

$$+ U[(-\frac{5}{6} + 3ac - 2a - 2a^2 - \frac{c^2}{3} - \frac{2c^3}{3})\sqrt{1-c^2} + (2ac + c - \frac{1}{2} - 2a - a^2)\cos^{-1}c]\beta'(t)$$

$$- (\frac{1}{2} + a^2 + a)\pi\ h''(t) - (\frac{1}{8} - \frac{a}{2} - a^3 - \frac{a^2}{2})\pi\alpha''(t)$$

$$- [(\frac{1}{8} + \frac{a^2}{2} - \frac{c}{2} - a^2c - ac)\cos^{-1}c + \sqrt{1-c^2}\ (\frac{1}{2} + a^2 + \frac{2}{3}a + \frac{1}{3}ac^2 - \frac{a^2c}{2} - \frac{c^3}{12} - \frac{c}{24})]\beta''(t)$$

$$+ \frac{(1-a)^2}{2}\ \Gamma'(t) + U \int_0^t (z - \sqrt{z^2-1} + a\frac{\sqrt{z+1}}{\sqrt{z-1}} - a)\Gamma'(t-\sigma)d\sigma$$

$$+ \frac{d}{dt} \int_0^t \frac{1}{2}[\frac{1}{2}\frac{\sqrt{z+1}}{\sqrt{z-1}} - z^2 + z\sqrt{z^2-1} + 2a(z - \sqrt{z^2-1}) + a^2(\frac{\sqrt{z+1}}{\sqrt{z-1}} - 1)]\Gamma'(t-\sigma)d\sigma\ \}$$

The last two lines may be written as

$$\frac{\Gamma'(t)}{2} + U \int_0^t c_4(t-\sigma)\ [B,Z(\sigma)]d\sigma + U\ a \int_0^t (1-c_2(t-\sigma))\ [B,\ (\sigma)]d\sigma$$

$$+ \frac{d}{dt}\ \{\ \frac{1}{4} \int_0^t (1-c_2(t-\sigma))[B,Z(\sigma)]d\sigma + \frac{1}{4}\ \Gamma(t)$$

$$+ \frac{1}{2} \int_0^t c_5(t-\sigma)[B,Z(\sigma)]d\sigma + a \int_0^t c_3(t-\sigma)[B,Z(\sigma)]d\sigma + \frac{a^2}{2}\ [B,Z(t)]\}$$

$$= \int_0^t [\tfrac{3}{4} c_1(t-\sigma) + U\, c_4(t-\sigma) + Ua(1-c_2(t-\sigma))][B, Z(\sigma)]d\sigma$$

$$+ \frac{d}{dt} \int_0^t [(\tfrac{1}{4} + \tfrac{a^2}{2}) - \tfrac{1}{4} c_2(t-\sigma) + a\, c_3(t-\sigma) + \tfrac{1}{2} c_5(t-\sigma)][B, Z(\sigma)]d\sigma$$

Where

$$c_4(t) = \int_0^t c_1(t - \sigma)\left(1 - U\sigma - \sqrt{U^2 \sigma^2 + 2U\sigma}\right) d\sigma$$

$$= c_2(t) + c_3(t)$$

$$c_5(t) = \int_0^t c_1(t - \sigma)\left((1 + U\sigma)\sqrt{U^2 \sigma^2 + 2U\sigma} - (1 + U\sigma)^2\right) d\sigma$$

Plots of $c_4(t)$ and $c_5(t)$ are given in Figures 5 and 6.

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CHAPTER SIX

CALCULATION OF THE FLAP MOMENT M_β

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6, Calculation of the Flap Moment M_β.

We have:

$$M_\beta = \int_c^1 (x-c)P(x,t)dx$$

$$= (-\rho)\ \{ \int_c^1 U(x-c)\gamma_a(t,x)dx + \frac{\partial}{\partial t} \int_c^1 (x-c)\ dx \int_{-1}^x \gamma_a(t,y)dy\}$$

$$= (-\rho)\ \{ \int_c^1 U(x-c)\gamma_a(t,x)dx + \frac{\partial}{\partial t} \frac{(1-c)^2}{2} \int_{-1}^1 \gamma_a(t,y)dy - \frac{\partial}{\partial t} \int_c^1 \frac{(x-c)^2}{2} \gamma_a(t,x)dx\}$$

$$= (-\rho)\ \{ U \int_c^1 (x-c)\gamma_a(t,x)dx + \frac{(1-c)^2}{2}\Gamma'(t) - \frac{1}{2} \frac{\partial}{\partial t} \int_c^1 (x-c)^2 \gamma_a(t,x)dx\}$$

Now

$$\int_{-1}^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{w_a(t,\zeta)}{x-\zeta} d\zeta$$

$$= \int_{-1}^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{(-h'(t) - (\zeta-a)\alpha'(t) - U\alpha(t))}{(x-\zeta)} d\zeta + \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{(-(\zeta-c)\beta'(t) - U\beta(t))d\zeta}{(x-\zeta)}$$

The first term

$$= + \pi\, h'(t) - \pi(a-1-x)\alpha'(t) + U\pi\alpha(t)$$

Next let

$$\frac{2}{\pi} \int_c^1 (x-c) \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, d\zeta = F_1(c)$$

$$\frac{2}{\pi} \int_c^1 (x-c) \frac{\sqrt{1-x}}{\sqrt{1+x}} \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\zeta}{x-\zeta}\, d\zeta = F_2(c)$$

Now let us define

$$H_1(\sigma) = \frac{2}{\pi} \int_c^1 (x-c) \frac{\sqrt{1-x}}{\sqrt{1+x}} H(\sigma,x)dx$$

$$= (-2) \int_c^1 \frac{(x-c)}{x-z} \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx \frac{\sqrt{z+1}}{\sqrt{z-1}}$$

$$= (-2) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left[\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} \left(1 + \frac{z-c}{x-z}\right)dx\right]$$

$$= (-2) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left[\cos^{-1}c - \sqrt{1-c^2} - (z-c)\cos^{-1}c + \frac{2(z-1)(z-c)}{\sqrt{z^2-1}} \operatorname{Tan}^{-1} \frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}}\right]$$

Hence

$$\int_c^1 (x-c)\gamma_a(t,x)dx$$

$$= \frac{2}{\pi} \int_c^1 (x-c) \frac{\sqrt{1-x}}{\sqrt{1+x}} (\pi h'(t) + \pi(x-1)\alpha'(t) + a\pi\alpha'(t) + U\pi\alpha(t))dx$$

$$- (F_2(c) - c\,F_1(c))\beta'(t) - U\,F_1(c)\beta(t) - \frac{1}{2\pi} \int_0^t H_1(\sigma)\Gamma'(t-\sigma)d\sigma$$

$$= 2(h'(t) - a\,\alpha'(t) + U\,\alpha(t))\left((1 + \frac{c}{2})\sqrt{1-c^2} - (c + \frac{1}{2})\cos^{-1}c\right)$$

$$+ \left[(\frac{2}{3} + \frac{c^2}{3})\sqrt{1-c^2} - \cos^{-1}c\right]\alpha'(t)$$

$$- (F_2(c)-c\,F_1(c))\beta'(t) - U\,F_1(c)\beta(t)$$

$$- \frac{1}{2\pi} \int_0^t H_1(\sigma)\Gamma'(t-\sigma)d\sigma$$

Next

$$\int_c^1 (x-c)^2 \gamma_a(t,x)dx$$

$$= \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} (\pi h'(t) - a\pi\alpha'(t) + U\pi\alpha(t))dx$$

$$+ 2 \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x+1)dx \;\; \alpha'(t)$$

$$- (g_2(c) - cg_1(c))\beta'(t) - U g_1(c)\beta(t) - \frac{1}{2\pi} \int_0^t H_2(\sigma)\Gamma'(t-\sigma)d\sigma$$

where $$H_2(\sigma) = \frac{2}{\pi}\int_c^1 (x - c)^2 \sqrt{\frac{1-x}{1+x}} \, H(\sigma, x)dx$$

$$g_1(c) = \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} d\zeta$$

$$g_2(c) = \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\zeta}{x-\zeta} d\zeta$$

Plots of $H_1(t)$ and $H_2(t)$ are given in Figures 7 and 8.

Now using formulas (23-24) from appendix 1, we have:

First Term = $2\, h_1(c)\, (h'(t) - a\alpha'(t) + U\alpha(t))$

Second Term = $+\, 2\, h_2(c)\alpha'(t)$

Next

$$H_2(\sigma) = \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}}\, H(\sigma,x)dx$$

$$= (-2) \left(\int_c^1 \frac{(x-c)^2}{(x-z)} \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx \right) \frac{\sqrt{z+1}}{\sqrt{z-1}}$$

$$= (-2) \frac{\sqrt{z+1}}{\sqrt{z-1}} \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} \left[\frac{(z-c)^2}{x-z} + x-2c+z\right]dx$$

$$= (-2) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left[(z-c)^2\left(-\cos^{-1}c + 2\frac{(z-1)}{\sqrt{z^2-1}} \operatorname{Tan}^{-1} \frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}}\right)\right.$$

$$\left. + (z-2c)(\cos^{-1}c - \sqrt{1-c^2}) + (\sqrt{1-c^2}) - \frac{1}{2} \cos^{-1}c - \frac{c\sqrt{1-c^2}}{2} \right]$$

$$= (-4)(z-c)^2 \operatorname{Tan}^{-1} \frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}} - 2\frac{\sqrt{z+1}}{\sqrt{z-1}} \left[(z-2c-(z-c)^2 - \frac{1}{2})\cos^{-1}c + (1+2c - \frac{c}{2} - z)\sqrt{1-c^2}\right]$$

Hence putting it altogether we have:

$$\frac{M_\beta}{\rho U^2} = U((2+c)\sqrt{1-c^2} - (2c+1)\cos^{-1}c)\,(h'(t) - a\,\alpha'(t) + U\,\alpha(t))$$

$$U[(\tfrac{2}{3} + \tfrac{c^2}{3})\sqrt{1-c^2} - c\,\cos^{-1}c]\alpha'(t)$$

$$- U\,(F_2(c) - c\,F_1(c))\beta'(t) - U^2\,F_1(c)\beta(t)$$

$$- \frac{U}{2\pi}\int_0^t H_1(\sigma)\Gamma'(t-\sigma)d\sigma + \frac{(1-c)^2}{2}\Gamma'(t) - h_1(c)\,(h''(t) - a\alpha''(t) + U\alpha'(t)) - h_2(c)\alpha''(t)$$

$$+ \left(\frac{g_2(c) - c\,g_1(c)}{2}\right)\beta''(t) + \frac{U}{2}\,g_1(c)\beta'(t) + \frac{1}{2}\left(\frac{1}{2\pi}\right)\frac{\partial}{\partial t}\int_0^t H_2(\sigma)\Gamma'(t-\sigma)d\sigma$$

$$= (-\rho)\;\{U^2[(2+c)\sqrt{1-c^2} - (2c+1)\cos^{-1}c]\alpha(t) - U^2\,f_1(c)\beta(t) + U\,[(2+c)\sqrt{1-c^2} - (2c+1)\cos^{-1}c]h'(t)$$

$$+ U\,[(\tfrac{8}{3} + c + \tfrac{c^2}{3})\sqrt{1-c^2} - (1+3c)\cos^{-1}c - h_1(c)]\alpha'(t) + U\,[\frac{g_1(c)}{2} - (f_2(c) - cf_1(c)]\beta'(t)$$

$$- h_1(c)h''(t) + (ah_1(c) - h_2(c))\alpha''(t) + \frac{1}{2}\,(g_2(c) - c\,g_1(c))\beta''(t) - \frac{U}{2\pi}\int_0^t H_1(\sigma)\Gamma'(t-\sigma)d\sigma$$

$$+ \frac{1}{2}\,\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t H_2(\sigma)\Gamma'(t-\sigma)d\sigma\;\}$$

The functions $f_1(c)$, $f_2(c)$, $g_1(c)$, $g_2(c)$ are calculated in Appendix II.

$$h_1(c) = \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)^2 dx$$

$$= \left(\frac{1+2c+2c^2}{2}\right) \cos^{-1} c - \frac{1}{6} (2c^2 + 9c + 4)\sqrt{1-c^2}$$

$$h_2(c) = \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x+1)(x-c)^2 dx$$

$$= \left(\frac{c^2}{2} + \frac{1}{8}\right) \cos^{-1} c - \left(\frac{c^3}{12} + \frac{13}{24}c\right) \sqrt{(1-c^2)}$$

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EQUATIONS OF MOTION

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7. Equations of Motion.

Let $x(t) = \begin{vmatrix} h(t) \\ \alpha(t) \\ \beta(t) \end{vmatrix}$

Then the typical section equations of motion with trailing-edge control can be written:

$$M_s\ddot{x} + B_s\dot{x} + K_s x = \frac{L}{m_s} + Gu$$

where the subscript s stands for structure, and

$$M_s = \begin{bmatrix} 1 & x_\alpha & x_\beta \\ x_\alpha & r_\alpha^2 & [r_\beta^2 + x_\beta(c-a)] \\ x_\beta & [r_\beta^2 + x_\beta(c-a)] & r_\beta^2 \end{bmatrix}$$

$$K_s = \begin{bmatrix} \omega_h^2 & 0 & 0 \\ 0 & r_\alpha^2\omega_\alpha^2 & 0 \\ 0 & 0 & r_\beta^2\omega_\beta^2 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2r_\beta^2\zeta_\beta\omega_\beta \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ r_\beta^2\omega_\beta^2 \end{bmatrix}$$

$$L = \begin{bmatrix} P \\ M_\alpha \\ M_\beta \end{bmatrix}$$

$$P = P_c + P_{nc}$$

$$M_\alpha = M_{\alpha,c} + M_{\alpha,nc}$$

$$M_\beta = M_{\beta,c} + M_{\beta,nc}$$

where the subscript c stands for 'circulatory' and nc for non-circulatory. Now we can write

$$\begin{bmatrix} P_{nc} \\ M_{\alpha,nc} \\ M_{\beta,nc} \end{bmatrix} = M_a\ddot{x} + B_a\dot{x} + K_a x$$

where the subscript 'a' stands for "aerodynamic"

$$K_a = [\frac{-\rho}{m_s}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & U^2(-1-2a)\pi & \pi U^2[(-1-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2}] \\ 0 & U^2[(2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c] & -U^2 f_1(c) \end{bmatrix}$$

Note the difference in the 2nd row corresponding to the pitching moment terms.

$$B_a = [\frac{-\rho}{m_s}] \begin{bmatrix} 0 & \pi U & U[c\sqrt{1-c^2} - \cos^{-1}c] \\ -\pi U(1+2a) & \pi U(a^2-a-\frac{1}{2}) & U\{(-\frac{5}{6} + 3ac - 2a - 2a^2 - \frac{c^2}{3} - \frac{2c^3}{3})\sqrt{1-c^2} \\ & & \quad - (\frac{1}{2} + a^2 + 2a - 2ac - c)\cos^{-1}c\} \\ U[(2+c)\sqrt{1-c^2} - (H_2c)\cos^{-1}c] & U[(\frac{8}{3} + c + \frac{c^2}{3})\sqrt{1-c^2} & U[\frac{g_1(c)}{2} - (f_2(c) - c\, f_1(c))] \\ & \quad - (1+3c)\cos^{-1}c - h_1(c)] & \end{bmatrix}$$

$$M_a = \left[\frac{-\rho}{m_s}\right] \begin{bmatrix} \pi & -a\pi & \sqrt{1-c^2} - c\cos^{-1}c - \frac{1}{3}(\sqrt{1-c^2})^3 \\ \pi\left(-\frac{1}{2} - a^2 - a\right) & \pi\left(a^3 + \frac{a^2}{2} + \frac{a}{2} - \frac{1}{8}\right) & -\left(\frac{1}{8} + \frac{a^2}{2} - \frac{c}{2} - a^2c - ac\right)\cos^{-1}c \\ & & -2\sqrt{1-c^2}\left(\frac{1}{4} + \frac{a^2}{2} + \frac{a}{3} + \frac{ac^2}{6} - \frac{a^2c}{4} - \frac{c^3}{24} - \frac{c}{48}\right) \\ -h_1(c) & a\,h_1(c) - h_2(c) & \frac{1}{2}(g_2(c) - c\,g_1(c)) \end{bmatrix}$$

Next let:

$$Z(t) = \text{Col.}\,(\,x(t),\ \dot{x}(t)\,).$$

Recall that:

$$B = \begin{vmatrix} 0 \\ 2\pi U \\ (2\cos^{-1}c + 2\sqrt{(1-c^2)})U \\ 2\pi \\ (1-2a)\pi \\ (1-2c)\cos^{-1}c + (2-c)\sqrt{(1-c^2)} \end{vmatrix}$$

Finally:

$$\begin{bmatrix} P_c \\ M_{\alpha,c} \\ M_{\beta,c} \end{bmatrix} \begin{matrix} = \\ = [\frac{-\rho}{m_s}] \\ = \end{matrix} \left[\begin{array}{|l} U \int_0^t c_2(t-\sigma)[B,\dot{Z}(\sigma)]d\sigma + U[B,z(0)] - \frac{d}{dt} \int_0^t c_3(t-\sigma)[B,\dot{Z}(\sigma)]d\sigma \\ \hline \int_0^t [\frac{3}{4} c_1(t-\sigma) + U c_4(t-\sigma) + U a(1-c_2(t-\sigma))][B,\dot{Z}(\sigma)]d\sigma \\ \quad + \frac{d}{dt} \int_0^t [(\frac{a^2}{2} + \frac{1}{4}) - \frac{1}{4} c_2(t-\sigma) + a\, c_3(t-\sigma) + \frac{1}{2} c_5(t-\sigma)][B,\dot{Z}(\sigma)]d\sigma \\ \hline - \frac{U}{2\pi} \int_0^t c_6(t-\sigma)[B,\dot{Z}(\sigma)]d\sigma + \frac{1}{2} \frac{d}{dt} \frac{1}{2\pi} \int_0^t c_7(t-\sigma)[B,\dot{Z}(\sigma)]d\sigma \end{array} \right]$$

$$= \int_0^t M_2(t-\sigma)\dot{Z}(\sigma)d\sigma + \frac{d}{dt} \int_0^t M_3(t-\sigma)\dot{Z}(\sigma)d\sigma$$

$$+ \begin{vmatrix} U[B,Z(0)] \\ 0 \\ 0 \end{vmatrix}$$

where

$$c_3(t) = \int_0^t c_1(t-\sigma)\,(U\sigma - \sqrt{2U\sigma + U^2\sigma^2}\,)d\sigma$$

$$c_4(t) = \int_0^t c_1(t-\sigma)\,(1 + U\sigma - \sqrt{U^2\sigma^2 + 2U\sigma})d\sigma$$

$$= c_2(t) + c_3(t)$$

$$c_5(t) = \int_0^t c_1(t-\sigma)\,((1 + U\sigma)\,\sqrt{U^2\sigma^2 + 2U\sigma} - (1 + U\sigma)^2)d\sigma$$

$$c_6(t) = \int_0^t c_1(t-\sigma)\,H_1(\sigma)d\sigma$$

$$c_7\,(t) = \int_0^t c_1(t-\sigma)\,H_2(\sigma)d\sigma$$

Plots of $c_6(t)$ and $c_7(t)$ are given in Figures 9 and 10.

The matrices $M_2(t)$ and $M_3(t)$ are

$$\underset{3\times 6}{M_2(t)} = \left(\frac{-\rho}{m_s}\right) \begin{bmatrix} \cup c_2(t) B^* \\ \left[\frac{3}{4} c_1(t) + \cup c_4(t) + a\cup(1-c_2(t))\right] B^* \\ \frac{-\cup}{2\pi} c_6(t)\, B^* \end{bmatrix}$$

$$\underset{3\times 6}{M_3(t)} = \left(\frac{-\rho}{m_s}\right) \begin{bmatrix} -c_3(t)\, B^* \\ \left[\left(\frac{a^2}{2} + \frac{1}{4}\right) - \frac{1}{4} c_2(t) + a c_3(t) + \frac{1}{2} c_5(t)\right] B^* \\ \frac{1}{4\pi} c_7(t)\, B^* \end{bmatrix}$$

The equations of motion can thus be written finally as:

$$M_s\ddot{x} + B_s\ddot{x} + K_s x = Gu + M_a\ddot{x} + B_a\ddot{x} + K_a x + \int_0^t M_2(t-\sigma)\dot{Z}(\sigma)d\sigma + \frac{d}{dt}\int_0^t M_3(t-\sigma)\dot{Z}(\sigma)d\sigma + \begin{bmatrix} \upsilon B^{*}Z(0) \\ 0 \\ 0 \end{bmatrix}\frac{(-\rho)}{m_s} \quad (6.1)$$

Or,

$$\dot{Z}(t) = AZ(t) + Hu(t) + [B,Z(0)]\begin{bmatrix} 0 \\ v \end{bmatrix} + \left[\int_0^t \begin{matrix} \bigcirc \\ (M_s-M_a)^{-1} \end{matrix} M_2(t-\sigma)\,\dot{Z}(\sigma)d\sigma\right] + \left[\frac{d}{dt} \begin{matrix} \bigcirc \\ (M_s-M_a)^{-1} \end{matrix} \int_0^t M_3(t-\sigma)\dot{Z}(\sigma)d\sigma\right]$$

Where

$$A = \begin{bmatrix} \bigcirc & I_{3x3} \\ -(M_s-M_a)^{-1}(K_s-K_a) & -(M_s-M_a)^{-1}(B_s-B_a) \end{bmatrix} ; \quad H = \begin{bmatrix} \bigcirc \\ (M_s-M_a)^{-1}G \end{bmatrix} ; \quad v = \left(\frac{-\rho}{m_s}\right)(M_s-M_a)^{-1}\begin{bmatrix} \upsilon \\ 0 \\ 0 \end{bmatrix}$$

Circulation: $\Gamma(t) = \int_0^t c_2(t-\sigma)\,[B,\dot{Z}(\sigma)]d\sigma + [B,Z(0)]$

Since $(M_s - M_a)^{-1} M_2(\infty) Z(0) = [B, Z(0)] \begin{bmatrix} 0 \\ v \end{bmatrix}$

and

$$\int_0^t M_2(t-\sigma)\,\dot{Z}(\sigma)\,d\sigma = M_2(\infty)\,Z(t) - M_2(\infty)\,Z(0) + \int_0^t \tilde{M}_2(t-\sigma)\,\dot{Z}(\sigma)\,d\sigma$$

we have finally:

$$\dot{Z} = \left(A + \begin{bmatrix} 0 \\ v \end{bmatrix} B^*\right) Z(t) + \left[\int_0^t (M_s - M_a)^{-1}\,\tilde{M}_2(t-\sigma) Z(\sigma)\,d\sigma\right] + \left[\frac{d}{dt}\,(M_s - M_a)^{-1} \int_0^t M_3(t-\sigma)\dot{Z}(\sigma)\,d\sigma\right]$$

$$+ H\,u(t) \qquad \ldots\ldots \quad (6.2)$$

where

$$\tilde{M}_2(t) = \begin{bmatrix} \bigcirc_{3\text{x}6} \\ U(c_2(t)-1)\,B^* \\ (\frac{3}{4}\,(c_1(t) + U\;c_4(t) + a\,U(1-c_2(t))\,B^* \\ -\frac{U}{2\pi}\,c_6(t)\,B^* \end{bmatrix} \left(\frac{-\rho}{m_s}\right)$$

Note also that

$$M_3(\infty) = \left(\frac{-\rho}{m_s}\right) \begin{bmatrix} \bigcirc_{3x6} \\ B^* \\ \left(\frac{a^2}{2} - a - \frac{1}{4}\right) B^* \\ \left[(4c + 1) \mathrm{Cos}^{-1} c - (2+3c) \sqrt{1-c^2} \right] B^* \end{bmatrix}$$

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CHAPTER EIGHT

STEADY STATE RESPONSE TO A STEP FLAP INPUT

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8. STEADY STATE RESPONSE TO STEP FLAP INPUT

In this section we shall calculate the steady state response to a step input deflection of the flap, assuming system stability.

8.1 Steady State Structural Response.

We begin with the calculation of the steady state structural response. The dynamic equations are:

$$M_s \ddot{x} + B_s \dot{x} + K_s x = Gu + M_a \ddot{x} + B_a \dot{x} + K_a x + M_2(\infty)\, Z(t)$$
$$+\int_0^t \tilde{M}_2(t-\sigma)\, \dot{Z}(\sigma) d\sigma$$
$$+\frac{d}{dt}\int_0^t M_3(t-\sigma)\, \dot{Z}(\sigma)\, d\sigma$$

$$\tilde{M}_2(t) = M_2(t) - M_2(\infty)$$

Hence the steady state response obtained by setting all time derivatives to zero and initial conditions to be zero, readily becomes:

$$(K_s - K_a)\, x(\infty) - \left(\frac{-\rho}{m_s}\right) U B^* Z(\infty) = G\, u(\infty)$$

In particular, taking $u(\infty) = 1$, we have:

$$\omega_h^2 \cdot h_\infty = \left(-\frac{\rho}{m_s}\right) U\, c_2(\infty)\ (2\pi U\alpha_\infty + (2\cos^{-1}c + 2\sqrt{1-c^2})U\beta_\infty)$$

$$r_\alpha^2\omega_\alpha^2 \cdot \alpha_\infty = \left(-\frac{\rho}{m_s}\right) [U^2\pi a\alpha_\infty - U^2(\cos^{-1}c - c\sqrt{1-c^2})]\beta_\infty$$

$$+ \left(-\frac{\rho}{m_s}\right) U\, c_2(\infty)\ [(-a)(2\pi U\alpha_\infty + (2\cos^{-1}c + 2\sqrt{1-c^2})U\beta_\infty)]$$

$$r_\beta^2\omega_\beta^2 \cdot \beta_\infty = r_\beta^2\omega_\beta^2 + \left[\frac{(-\rho)}{m_s}\right] [U^2\ ((2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c)\alpha_\infty - U^2 f_1(c)\beta_\infty].$$

Or, finally:

$$\omega_h^2 \cdot h_\infty - \left(\frac{-\rho}{m_s}\right) U\, c_2(\infty)\ (2\pi U)\alpha_\infty - \left(\frac{-\rho}{m_s}\right) U^2 c_2(\infty)(2\cos^{-1}c + 2\sqrt{1-c^2})\beta_\infty = 0$$

$$0 \cdot h_\infty + [-r_\alpha^2\omega_\alpha^2 + \left(\frac{\pm\rho}{m_s}\right)U^2\pi a + a\left(\frac{+\rho}{m_s}\right) c_2(\infty)2\pi U^2]\alpha_\infty$$

$$+ [(+a)\left(\frac{+\rho}{m_s}\right) U^2 c_2(\infty)\ (2\cos^{-1}c + 2\sqrt{1-c^2})$$

$$+ \left(\frac{+\rho}{m_s}\right)U^2(\cos^{-1}c - c\sqrt{1-c^2})]\ \beta_\infty = 0$$

$$0 \cdot h + \left(\frac{+\rho}{m_s}\right) U^2[(2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c)]\alpha_\infty$$

$$+ \{[\frac{+\rho}{m_s}]\ [-U^2 f_1(c)] + r_\beta^2\omega_\beta^2\}\beta_\infty = r_\beta^2\omega_\beta^2$$

Letting

$$M_\infty = \begin{bmatrix} \omega_h^2 & \left(\frac{\rho}{m_s}\right) U^2 \cdot 2\pi & \left(\frac{\rho}{m_s}\right) U^2 \left(2\cos^{-1}c + 2\sqrt{1-c^2}\right) \\ 0 & r_\alpha^2\omega_\alpha^2 - \left(\frac{\rho}{m_s}\right) U^2 (2a\pi+\pi) & \left(\frac{\rho}{m_s}\right) U^2 \left[(c-2a)\sqrt{1-c^2} - (1+2a)\cos^{-1}c\right] \\ 0 & \left(\frac{\rho}{m_s}\right) U^2 \left[(2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c\right] & \begin{array}{l}\left(\frac{\rho}{m_s}\right) \frac{U^2}{\pi} \left[2\sqrt{1-c^2}\cos^{-1}c + (1-c^2)\right. \\ \qquad \left. - (1+2c)(\cos^{-1}c)^2\right] \\ \qquad + r_\beta^2\omega_\beta^2 \end{array} \end{bmatrix}$$

We have, for the steady-state values in general,

$$M_\infty \, x(\infty) = G \, u(\infty)$$

This yields for the data of Table 1

$$\begin{bmatrix} (2.5)10^3 & 0 & 0 \\ 0 & (2.07)10^3 & (6.25)10^2 \\ 0 & (2.6)10 & (6.11)10^2 \end{bmatrix} \begin{bmatrix} h_\infty \\ \alpha_\infty \\ \beta_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5.625 \times 10^2 \end{bmatrix}.$$

Or

$$\begin{bmatrix} h_\infty \\ \alpha_\infty \\ \beta_\infty \end{bmatrix} = \begin{bmatrix} (4.0)10^2 & (-7.9)10^{-4} & (-6.9)10^{-4} \\ 0 & (4.87)10^{-4} & (-4.98)10^{-4} \\ 0 & (-2.12)10^{-5} & (1.65)10^{-3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (5.625)10^2 \end{bmatrix} = \begin{bmatrix} -.39 \\ -.28 \\ 0.93 \end{bmatrix}$$

Steady State Pressure, Lift and Moments

We can also calculate the steady state pressure distribution due to flap deflection directly from the aerodynamic equations. We have

Steady State Downwash

$$w_a(\infty,x) = -U\alpha_\infty, \quad -1 < x < c$$

$$= -U\alpha_\infty - U\beta_\infty, \quad c < x < 1$$

Steady State Circulation on the Foil

$$\gamma_a(\infty,x) = \frac{2}{\pi}\frac{\sqrt{1-x}}{\sqrt{1+x}} \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{w_a(\infty,\zeta)}{x-\zeta}\, d\zeta$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}}(2U\alpha) - (2U\beta)\frac{1}{\pi}\frac{\sqrt{1-x}}{\sqrt{1+x}} \int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta}\, d\zeta$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}}(2U\alpha) + \frac{(2U\beta)}{\pi}\frac{\sqrt{1-x}}{\sqrt{1+x}}\left\{ \cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \operatorname{Log}\left| \frac{\sqrt{(1-c)}\sqrt{1+x} + \sqrt{(1+c)}\sqrt{1-x}}{\sqrt{(1-c)}\sqrt{1+x} - \sqrt{(1+c)}\sqrt{1-x}} \right| \right\}$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}}(2U\alpha) + \frac{(2U\beta)}{\pi}\frac{\sqrt{1-x}}{\sqrt{1+x}}\cos^{-1}c + \frac{(2U\beta)}{\pi}\operatorname{Log}\left| \frac{\sqrt{(1-c)}\sqrt{1+x} + \sqrt{(1+c)}\sqrt{1-x}}{\sqrt{(1-c)}\sqrt{1+x} - \sqrt{(1+c)}\sqrt{1-x}} \right|$$

Note that $\gamma_a(\infty,x)$ is discontinuous at $x = c$. Note also that

$$\gamma_a(\infty, 1) = 0 = \Gamma'(\infty)$$

Figure 11 is a plot of the circulation, $\gamma_a(\infty, x)$, on the airfoil due to a unit step change in flap deflection.

Steady State Pressure

$$P(\infty,x) = (-\rho)\ U\ \gamma_a(\infty,x)$$

Steady State Lift

$$L = \int_{-1}^{1} P(\infty,x)dx$$

$$= (-\rho)\ U \int_{-1}^{1} \gamma_a(\infty,x)dx$$

$$= (-\rho)\ U^2\{\ 2\pi\alpha + 2\beta\ \cos^{-1}c + \frac{2\beta}{\pi} \int_{-1}^{1} x\ \frac{\sqrt{1-c^2}}{x-c}\ \frac{1}{\sqrt{1-x^2}}\ dx\ \}$$

$$= (-\rho)\ U^2\ \{2\pi\alpha + 2\beta\ \cos^{-1}c + (2\beta)\ \sqrt{1-c^2}\ \}$$

Steady State Moment M_α

$$M_\alpha = (-\rho)\ U \int_{-1}^{1} (x-a)\ \gamma_a(\infty,x)dx$$

$$= (-\rho)\ U^2\ \{[(-\pi\ -2a\pi)\alpha\ -\ ((1+2a)\cos^{-1}c]\beta\ + \frac{2\beta}{\pi} \int_{-1}^{1} \frac{(x-a)^2}{2}\ \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{1}{x-c}\ dx\}$$

$$= (-\rho)U^2\ \{-(2a+1)\pi\alpha + \beta\ [-(2a+1)\cos^{-1}c + (c-2a)\ \sqrt{1-c^2}]\}$$

Steady State Moment M_β

$$M_\beta = (-\rho)U \int_c^1 (x-c)\ \gamma_a(\infty,x)dx$$

$$= (-\rho)\ U^2\ \{[(2+c)\ \sqrt{1-c^2} - (2c+1)\ \cos^{-1}c]\alpha$$

$$+ [\ \frac{1}{\pi}(2+c)\ \sqrt{1-c^2} - \frac{1}{\pi}(2c+1)\cos^{-1}c]\ (\cos^{-1}c)]\beta$$

$$+ \frac{\beta}{\pi} \int_c^1 (x-c)^2\ \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{1}{x-c}\ dx\}$$

$$= (-\rho)U^2\{\ [(c+2)\sqrt{1-c^2} - (1+2c)\ \cos^{-1}c]\alpha$$

$$+ (\frac{1}{\pi})\ [2\sqrt{1-c^2}\ \cos^{-1}c - (1+2c)\ (\cos^{-1}c)^2 + (1-c^2)]\beta\}$$

Finally noting that the steady state response to step input (unit flap deflection) is determined by:

$$K_s x_\infty = \frac{L}{m_s} + Gu$$

we obtain

$$\begin{bmatrix} \omega_h^2 & 0 & 0 \\ 0 & r_\alpha^2\omega_\alpha^2 & 0 \\ 0 & 0 & r_\beta^2\omega_\beta^2 \end{bmatrix} x_\infty = \begin{vmatrix} \frac{P}{m_s} \\ \frac{M_\alpha}{m_s} \\ \frac{M_\beta}{m_s} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ r_\beta^2\omega_\beta^2 \end{vmatrix}$$

leading to the same equations as before.

In particular, for the data of Table 1, we have:

$$\begin{bmatrix} \dfrac{P}{m_s} \\[2ex] \dfrac{M_\alpha}{m_s} \\[2ex] \dfrac{M_\beta}{m_s} \end{bmatrix} = \left| k_s x_\infty - Gu \right| = \begin{vmatrix} \omega_h^2 \cdot h_\infty \\[2ex] r_\alpha^2 \omega_\alpha^2 \alpha_\infty \\[2ex] r_\beta^2 \omega_\beta^2 (\beta_\infty - 1) \end{vmatrix} = \begin{vmatrix} -975.00 \\[2ex] -700 \\[2ex] -38.25 \end{vmatrix}$$

A plot of the steady state circulation on the foil is given in figure 2. Note the logarithmic discontinuity at the hinge point x=c. Note also that the Kutta condition is clearly satisfied at the trailing edge.

CHAPTER NINE

CALCULATION OF THE TRANSIENT RESPONSE

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9. CALCULATION OF TRANSIENT RESPONSE.

In this section we indicate our technique for calculating the transient response of the three-degree-of-freedom typical section to a step input to the flap.

For this purpose it is convenient to rewrite the basic dynamic equations in the 'integrated' form, and taking $Z(0+)$ to be zero, without much loss in generality.

Integrating the equations of motion:

$$Z(t) = \int_0^t AZ(\sigma)\, d\sigma + \left[\int_0^t (M_s - M_a)^{-1} M_2(t-\sigma) Z(\sigma)\, d\sigma\right] + \left[(M_s - M_a)^{-1} \int_0^t M_3(t-\sigma)\dot{Z}(\sigma)\, d\sigma\right] + Ht$$

We rewrite this equation in terms of $\dot{Z}(t)$:

$$\int_0^t \dot{Z}(\sigma)d\sigma - \int_0^t A\dot{Z}(\sigma)(t-\sigma)d\sigma - \left[\int_0^t (M_s - M_a)^{-1} M_4(t-\sigma)\dot{Z}(\sigma)d\sigma\right] - \left[(M_s - M_a)^{-1} \int_0^t M_3(t-\sigma)\dot{Z}(\sigma)d\sigma\right] = Ht$$

where

$$M_4(t) = \int_0^t M_2(\sigma)d\sigma.$$

Hence, we need to solve the convolution integral equation for $\dot{Z}(t)$:

$$\int_0^t M(t-\sigma)y(\sigma)d\sigma = Ht \quad ; \quad (y(\sigma) = \dot{Z}(\sigma))$$

where

$$M(t) = I - At - \begin{bmatrix} \bigcirc \\ (M_s - M_a)^{-1} M_4(t) \end{bmatrix} - \begin{bmatrix} \bigcirc \\ (M_s - M_a)^{-1} M_3(t) \end{bmatrix}$$

$$H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (M_s - M_a)^{-1} G \end{bmatrix} \quad ; \qquad G = \begin{bmatrix} 0 \\ 0 \\ r_\beta^2 \, \omega_\beta^2 \end{bmatrix}$$

$$M_2(t) = \left(-\frac{\rho}{m_s}\right) \begin{bmatrix} \upsilon c_2(t)B^* \\ \left[\frac{3}{4} c_1(t) + \upsilon c_4(t) + a\upsilon(1-c_1(t)\right] B^* \\ -\frac{\upsilon}{2\pi} c_6(t)B^* \end{bmatrix}$$

$$M_3(t) = \left(-\frac{\rho}{m_s}\right) \begin{bmatrix} -c_3(t)B^* \\ \left[\left(\frac{a^2}{2} + \frac{1}{4}\right) - \frac{1}{4} c_2(t) + ac_3(t) + \frac{1}{2} c_5(t)\right] B^* \\ \frac{1}{4\pi} c_7(t)B^* \end{bmatrix}$$

$$\int_0^\infty e^{-st} c_1(t)dt = \frac{1}{s\int_0^\infty e^{-s\sigma} \frac{\sqrt{2+\upsilon\sigma}}{\sqrt{\upsilon\sigma}} d\sigma}$$

$$c_1(t) = \frac{\sqrt{2}}{\pi} \upsilon \left\{ \frac{1}{2} (\upsilon t)^{-\frac{1}{2}} - \frac{1}{8} (\upsilon t)^{\frac{1}{2}} + \frac{5}{192} (\upsilon t)^{3/2} \ldots \right\} \quad , \quad \upsilon t < 5$$

$$c_2(t) = \int_0^t c_1 d\sigma$$

$$c_3(t) = \int_0^t c_1(t-\sigma)(\upsilon\sigma - \sqrt{\upsilon^2\sigma^2 + 2\upsilon\sigma})d\sigma$$

$$c_4(t) = c_2(t) + c_3(t)$$

$$c_5(t) = \int_0^t c_1(t-\sigma) \quad [(1 + \upsilon\sigma) \sqrt{\upsilon^2\sigma^2 + 2\upsilon\sigma} - (1+\upsilon\sigma)^2]d\sigma$$

$$c_6(t) = \int_0^t c_1(t-\sigma) H_1(\sigma)d\sigma$$

$$c_7(t) = \int_0^t c_1(t-\sigma) H_2(\sigma)d\sigma$$

$$H_1(t) = (-2)\,\frac{\sqrt{z+1}}{\sqrt{z-1}}\left[\cos^{-1}c - \sqrt{1-c^2} - (z-c)\cos^{-1}c + 2\frac{\sqrt{z-1}}{\sqrt{z+1}}\,(z-c)\,\tan^{-1}\frac{\sqrt{(1-c)(1+z)}}{\sqrt{(1+c)(z-1)}}\right], \quad z = 1 + \upsilon t$$

$$H_2(t) = (-4)\,(z-c)^2\,\tan^{-1}\frac{\sqrt{(1-c)(z+1)}}{\sqrt{(1+c)(z-1)}} - 2\frac{\sqrt{z+1}}{\sqrt{z-1}}\left[(z - 2c - (z-c)^2 - \frac{1}{2})\cos^{-1}c + (1 + 2c - \frac{c}{2} - z)\,\sqrt{1-c^2}\right],$$

$$z = 1 + \upsilon t$$

$$B^* = \left[0 \cdot 2\pi\upsilon \cdot (2\cos^{-1}c + 2\,\sqrt{1-c^2})\,\upsilon \cdot (2\pi) \quad \cdot \quad (\pi - 2\pi a) \quad \cdot \quad (2-c)\,\sqrt{1-c^2} + (1-2c)\cos^{-1}c\right]$$

Solution for zero stream velocity : $U = 0$

It is of interest to examine the limiting case when $U=0$. We have:

$$M_2(t;\, U=0) = 0$$

$$M_3(t; U=0) = (-\rho / m_s) \left| \begin{array}{c} 0 \\ ((a^2/2 + \tfrac{1}{4})\, B^* \\ (1/4\pi)\left[(1+2c^2)\cos^{-1} c - 3c\sqrt{(1-c^2)}\right] B^* \end{array} \right|$$

(Let us denote the matrix on the right by $M_{3,0}$)

$$K_a = 0$$

$$B_a = 0.$$

Thus the dynamic equations for the case when $U=0$ can be written:

$$\dot{Z}(t) = A\, Z(t) + H\, u(t) + \left| \begin{array}{c} 0 \\ (M_s - M_a)^{-1} M_{3,0} \end{array} \right| \dot{Z}(t)$$

from which we can readily calculate the in vacuo modes. These modes and corresponding eigen vectors are given in Table 2 for data set #1.

Solution at $t = 0+$

We next calculate the transient solution for small t, specifically for $t < 1/U$. For such values of t, we can exploit the approximation:

$$M_2(t) \approx (-\rho/m_s) \begin{vmatrix} 0 \\ (3/8)(\sqrt{2}/\pi)\, U\, (Ut)^{-\frac{1}{2}}\, B^* \\ (U/2\pi)\sqrt{2}\,(c \cos^{-1} c - \sqrt{(1-c^2)})\, B^* \end{vmatrix}$$

$$M_3(t) \approx (-\rho/m_s) \begin{vmatrix} 0 \\ \tfrac{1}{4}\,(2a^2 + 1)\, B^* \\ (1/4\pi)\left[(1+2c^2)\cos^{-1} c - 3c\sqrt{(1-c^2)}\right] B^* \end{vmatrix}$$

$$M_4(t) \approx 0$$

We note that:

$$M_3(0+) = M_{3,0}\,.$$

Unlike the case for $U=0$, however, it should be noted that K_a and B_a are no longer zero.

The Solution for large t : Steady State Solution

Let us next consider the steady state solution. For $t \to \infty$, we can make the approximation:

$$M(t) \approx I - \left(A + \begin{vmatrix} 0 \\ (M_s - M_a)^{-1} M_2(\infty) \end{vmatrix} \right) t - \begin{vmatrix} 0 \\ (M_s - M_a)^{-1} M_3(\infty) \end{vmatrix}$$

Letting:

$$A_2 = A + \begin{vmatrix} 0 \\ (M_s - M_a)^{-1} M_2(\infty) \end{vmatrix}$$

$$A_3 = \begin{vmatrix} 0 \\ (M_s - M_a)^{-1} M_3(\infty) \end{vmatrix}$$

the asymptotic solution at $t = \infty$, is given by:

$$(I - A_3)\dot{Z}(t) = A_2 Z(t) + H u(t) + [B, Z(0)] \begin{vmatrix} 0 \\ v \end{vmatrix}$$

We remark that this solution is 'exact' for U = 0. A question that naturally arises is whether the matrix $(I-A_3)$ can be singular, in which we case we have a singular differential equation. It is nonsingular for U=0, as we have already seen.

NUMERICAL SOLUTION TECHNIQUE

For the numerical computation of the transient response due to step input, we proceed as follows. We take $Z(0)$ to be zero. We choose a sufficiently small sampling interval Δ , and proceed to discretise as usual. Some care has to be taken because of the singularity of the kernel $M_3(t)$.

First we note that the initial condition can be calculated from:

$M(0)\ y(0+) = H$ (as is evident by Laplace Transforming or otherwise)

$$y(0+) = M(0)^{-1} H \qquad ; \qquad M(0) = I - \begin{bmatrix} 0 \\ (M_s - M_a)^{-1} M_3(0) \end{bmatrix}$$

Next: use the discretization:

$$\sum_{0}^{n-1} \left(\int_{k\Delta}^{(k+1)\Delta} M(n\Delta-\sigma)d\sigma \right) y(k\Delta) \left[\text{or} \left(\frac{y(\overline{k+1}\Delta) + y(k\Delta)}{2} \right) \right] = H\ (n\Delta)\ , n \geq 1$$

Let

$$M_{n-k-1} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} M(n\Delta-\sigma)d\sigma$$

[In all kernels except $M_3(\cdot)$, we replace the right-hand side by the integrand at the upper limit.]

Then we have:

$$\sum_{0}^{n-1} M_{n-k-1}\, y(k\Delta) = nH$$

Since the matrix involved is "triangular", this can be solved readily without resort to iteration.

Note that the "coefficient" of

$$y(\overline{n-1}\Delta) \left[\text{or} \left(\frac{y(n\Delta) + y(\overline{n-1}\Delta)}{2} \right) \right] \text{ is always } \frac{1}{\Delta} \int_0^{\Delta} M(\Delta-\sigma)d\sigma \doteq M(0+).$$

The transient response of the section has been calculated for the section parameters of Table 1 and for nondimensional velocities of 200, 225, 250, and 275. The responses are given in Figure 12.

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CALCULATION OF THE GUST RESPONSE

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10.1 CALCULATION OF CIRCULATION DUE TO GUST

We begin by calculating the circulation due to gust. First, we let $g(x)$, $-\infty < x < \infty$, represent the "frozen" gust velocity (field). This is a spatially homogeneous random process, Gaussian-distributed with spectral density $p(\lambda)$ which can (following the Von Karman theory) be taken as:

$$p(\lambda) = \text{Const.}\ \frac{1 + k_1\lambda^2}{(1 + k_2\lambda^2)^{11/6}}$$

where k_1 and k_2 are constants.

From unpublished calculations, $\gamma_g(t,x)$, the circulation on the foil, is given by the integral equation:

$$\gamma_g(t,x) = \frac{+2}{\pi} \frac{\sqrt{1 - x}}{\sqrt{1 + x}} \left\{ - \int_{-1}^{1} \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} \frac{g(\zeta - Ut)}{x - \zeta} d\zeta - \frac{1}{2\pi} \int_0^t H(\sigma,x)\, \Gamma_g'(t - \sigma) d\sigma \right\}$$

where $\Gamma_g(t)$ is the total circulation due to gust, defined by:

$$\Gamma_g(t) = \int_{-1}^{1} \gamma_g(t,x) dx$$

and

$$H(\sigma,x) = \frac{-\pi}{x - z}\frac{\sqrt{z + 1}}{\sqrt{z - 1}} \quad , \quad z = 1 + U\sigma$$

It is convenient to use the spectral representation for the gust field:

$$g(x - Ut) = \int_{-\infty}^{\infty} e^{2\pi i\, \nu(x - Ut)}\, dG(\nu)$$

where

$$E\,|dG(\nu)|^2 = p(\nu)d\nu$$

Using the fact that the order of integration in

$$\int_{-1}^{1} \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\, dx \int_{-1}^{1} \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}}\; \frac{g(\zeta - t)}{x - \zeta}\, d\zeta$$

can be reversed, we obtain upon integrating in x over $[-1,+1]$:

$$\Gamma_g(t) = +2 \int_{-1}^{1} \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}}\, g(\zeta - t)\, d\zeta - \frac{1}{2\pi} \int_0^t H(\sigma)\, f_g'(t - \sigma)\, d\sigma$$

where

$$H(\sigma) = 2\pi \quad \frac{\sqrt{z+1}}{\sqrt{z-1}} - 1$$

The equation can be simplified to:

$$\frac{1}{2\pi} \int_0^t \frac{\sqrt{2 + U\sigma}}{\sqrt{U\sigma}} \Gamma_g' (t - \sigma)\, d\sigma = 2 \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, g(\zeta - Ut)\, d\zeta$$

This shows that the circulation is asymptotically stationary Gaussian. We can calculate the spectral density in the following way.

$$+2 \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \int_{-\infty}^{\infty} e^{2\pi i\nu(\zeta - Ut)}\, dG(\nu)\, d\zeta = (+2) \int_{-\infty}^{\infty} e^{-2\pi i\nu Ut} \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} e^{2\pi i\nu\zeta}\, d\zeta$$

Now

$$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} e^{2\pi i\nu\zeta}\, d\zeta = \int_{-1}^{1} \frac{1}{\sqrt{1-\zeta^2}} e^{2\pi i\nu\zeta}\, d\zeta - \frac{1}{(2\pi i)} \frac{d}{d\nu} \int_{-1}^{1} \frac{1}{\sqrt{1-\zeta^2}} e^{2\pi i\nu\zeta}\, d\zeta$$

$$= \pi J_0(2\pi\nu) + \frac{\pi}{i} J_1(2\pi\nu)$$

$$= \pi [J_0(2\pi\nu) + i J_1(2\pi\nu)]$$

where we have used:

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

$$J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos t\,x\,dt}{\sqrt{1 - t^2}}$$

Hence,

$$S(t) = 2 \int_{-1}^{1} \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}}\, g(\zeta - Ut)\, d\zeta$$

$$= (2\pi) \int_{-\infty}^{\infty} (J_0(2\pi\nu) + i\, J_1(2\pi\nu))\, e^{-2\pi i \nu U t}\, dG(\nu)$$

Now, substituting for $H(\sigma)$, we get:

$$\int_0^t \frac{\sqrt{2 + U\sigma}}{\sqrt{U\sigma}}\, \Gamma'_g(t - \sigma)\, d\sigma = S(t)$$

where $S(Ut)$ is a stationary stochastic process with spectral density

$$4\pi^2 \left|J_0(2\pi\nu) + i\, J_1(2\pi\nu)\right|^2 p(\nu)$$

Hence $\Gamma'_g(t)$ and $\Gamma_g(t)$ are also asymptotically stationary, and $g(t)$ has the spectral density

$$P_{\Gamma_g}(\nu) = \frac{4\pi^2 \left|J_0\left(\frac{2\pi\nu}{U}\right) + i\, J_1\left(\frac{2\pi\nu}{U}\right)\right|^2 P(\nu/U)}{|L(\nu)|^2}\left(\frac{1}{U^2}\right)$$

where

$$L(\nu) = \frac{1}{2\pi}\int_0^{\infty} e^{-2\pi i\nu t}\,\frac{\sqrt{1+Ut}}{\sqrt{Ut}}\,dt$$

$$= \frac{1}{2\pi U}\, e^{2\pi i\nu/U}\, \left(K_0\left(\frac{2\pi\nu}{U}\right) - K_0'(2\pi\nu/U)\right)$$

where

$$K_0(s) = \int_1^{\infty} e^{-st}\,\frac{dt}{\sqrt{t^2-1}}$$

Putting $\lambda = \nu U$, we can write

$$P_{\Gamma_g}(U\lambda) = (16\pi^4)\,\frac{\left|J_0(2\pi\lambda) + i\,J_1(2\pi\lambda)\right|^2}{\left|K_0(2\pi\lambda) - K_0'(2\pi\lambda)\right|^2}\,p(\lambda)$$

The main result is that asymptotically (large t) the circulation is a stationary Gaussian Random Process with the above spectral density. The Spectral density depends on the assumed density for the frozen gust field. We note that the latter is well approximated by the Dryden form:

$$p(\lambda) = \text{Const.}\ \frac{1}{1 + k_2\lambda^2}$$

especially at high frequencies. We obtain this approximation by noting that

$$\frac{1 + k_1\lambda^2}{(1 + k_2\lambda^2)^{11/6}} = \frac{(1 + k_2\lambda^2)}{(1 + k_2\lambda^2)^{11/6}} + \frac{(k_2 - k_1)\lambda^2}{(1 + k_2\lambda^2)^{11/6}}$$

where the second term goes to zero for large λ and

$$\frac{1 + k_2\lambda^2}{(1 + k_2\lambda^2)^{11/6}} \sim \frac{1}{1 + k_2\lambda^2}$$

we can also get a "time-domain" representation for the gust circulation:

$$\Gamma'_g(t) = (2\pi) \int_0^t c_1(t - \sigma)\ S(\sigma)\ d\sigma$$

where $c_1(t)$ is as defined above and has the Laplace transform:

$$\int_0^\infty e^{-st}\, c_1(t)\, dt = \frac{U}{s}\, e^{-st}\, [k_0(\tfrac{s}{U}) - k_0'(\tfrac{s}{U})]^{-1}$$

where $k_0(\cdot)$ is the modified Bessel function of order zero. We note that $\Gamma_g'(t)$ is a stationary Gaussian process for all $t > 0$, not just asymptotically. The circulation is asymptotically stationary, since

$$\Gamma_g(t) = \int_0^t \Gamma_g'(\sigma)\, d\sigma + \Gamma_g(0) = \Gamma_g(0) + \int_0^t c_2(t - \sigma) S(\sigma)\, d\sigma$$

where the natural choice for $\Gamma_g(0)$ would be zero.

10.2 CALCULATION OF THE LIFT DUE TO GUST LOADING

Having determined $\Gamma_g(t)$, we can go on to calculate the lift P_g. Thus, we have in the usual way:

$$P_g \doteq (-\rho)\, [U \quad g(t) - \frac{d}{dt} \int_{-1}^{1} x\, \gamma_g(t,x)\, dx + \Gamma_g{}'(t)$$

We need to calculate only:

$$\int_{-1}^{1} x\, \gamma_g(t,x)\, dx$$

which, using the integral equation for $\gamma_g(t,x)$, yields:

$$= +\frac{2}{\pi} \int_{-1}^{1} x \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{dx}{x-\zeta} \int_{-1}^{1} \frac{1+\zeta}{1-\zeta}\, g(\zeta - Ut)d\zeta - \frac{1}{2\pi} \int_{0}^{t} \int_{-1}^{1} x\, H(\sigma,x)\, dx \quad \Gamma_g{}'(t-\sigma)\, d\sigma$$

Now it can be shown that:

$$\int_{-1}^{1} x\, \gamma_g(t,x)dx = \frac{d}{dt} \quad (+2) \int_{-1}^{1} (1-\zeta) \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, g(\zeta - Ut)d\zeta \quad + \Gamma_g{}'(t) + \frac{d}{dt} \int_{0}^{t} (U\sigma - \sqrt{U^2\sigma^2 + 2U\sigma})\Gamma'(t-\sigma)d\sigma$$

Hence,

$$P_g(t) = (-\rho)[U\Gamma_g(t) - 2\,\frac{d}{dt}\int_{-1}^{1} (1 - \zeta^2)\, g(\zeta - Ut)d\zeta - \frac{d}{dt}\int_0^t (U\sigma - \sqrt{U^2\sigma^2 + U\sigma})\, \Gamma_g{}'(t - \sigma)d\sigma]$$

It follows that $P_g(t)$ is a Gaussian process with the "time-domain" representation:

$$P_g(t) = (-\dot{\rho})[U\Gamma_g(t) - \frac{d}{dt}\int_0^t c_3(\sigma)\, S(t - \sigma)d\sigma + \frac{d}{dt} S_2(t)]$$

where

$$S_2(t) = -2\int_{-1}^{1} \sqrt{1 - \zeta^2}\;\; g(\zeta - Ut)d\zeta$$

and is a stationary Gaussian process defined by

$$S_2(t) = -2\int_{-\infty}^{\infty} e^{-2\pi i\nu Ut} \left(\int_{-1}^{1} \sqrt{1 - \zeta^2}\;\; e^{2\pi i\nu\zeta}\right) d\zeta \;\; dG(\nu)$$

$$= -\pi\int_{-\infty}^{\infty} e^{-2\pi i\nu Ut}\, (J_0(2\pi\nu) + J_2(2\pi\nu))\, dG(\nu)$$

Here we have used:

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

$$= J_0(x) - \frac{[J_0(x) + J_2(x)]}{2}$$

$$= \frac{J_0(x) - J_2(x)}{2}$$

and

$$\int_{-1}^{1} \sqrt{1 - \zeta^2}\, e^{2\pi i\nu\zeta}\, d\zeta = \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, e^{2\pi i\nu\zeta}\, d\zeta - \frac{1}{(2\pi i)} \frac{d}{d\nu} \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, e^{2\pi i\nu\zeta}\, d\zeta$$

$$= \pi\, [J_0(2\pi\nu) + i\, J_1(2\pi\nu)] - \frac{1}{2i} [(2\pi)\ (J_0'(2\pi\nu) + i\, J_1'(2\pi\nu)]$$

$$= \pi\, [J_0(2\pi\nu) - \frac{J_0(2\pi\nu) - J_2(2\pi\nu)}{2}$$

$$= \frac{\pi}{2} [J_0(2\pi\nu) + J_2(2\pi\nu)]$$

We also note that for large ν

$$J_m(2\pi\nu) + J_0(2\pi\nu) \sim \frac{1}{\sqrt{2\pi\nu}}$$

Note that

$$\int_0^\infty e^{-st} [t - \sqrt{t^2 + 2t}\,]\, dt = \frac{1}{s^2} - \int_1^\infty e^{-st} \sqrt{t^2 - 1}\, dt$$

$$= \frac{1}{s^2} + \frac{e^s}{s} K_0'(s)$$

$$= \frac{1}{s^2} - \frac{e^s}{s} K_1(s)$$

$$= \frac{1}{s^2} [1 - se \quad K_1(s)$$

and hence, that

$$\int_0^\infty e^{-st} c_3(t)dt = \frac{U^2}{s^3} e^{-s/U} [1 - \frac{s}{U} e^{s/U} K_1(s/U)] / [K_0(s/U) + K_1(s/U)]$$

A time-domain representation for P_g would be, (taking $\Gamma_g(0) = 0$)

$$P_g(t) = (-\rho)\left\{\int_0^t (Uc_2(\sigma) - c_3'(\sigma))\, S(t-\sigma)\,d\sigma + \frac{d}{dt} S_2(t)\right\}$$

The corresponding spectral density (of the asymptotically stationary process) is

$$= \left|[(\text{Fourier Transform of } (Uc_2(t) - c_3'(t))(1/U)(J_0(\tfrac{2\pi\nu}{U}) + J_1(\tfrac{2\pi\nu}{U})) - \frac{(2\pi i\nu)}{U}(J_0(\tfrac{2\pi\nu}{U}) + J_2(\tfrac{2\pi\nu}{U}))]\right|^2 p\left(\frac{\nu}{U}\right)$$

Now, the Laplace Transform of $[Uc_2(t) - c_3'(t)]$ is

$$= \left(\frac{U}{s}\right) \cdot \frac{K_1(s/U)}{[K_0(s/U) + K_1(s/U)]}$$

Hence, spectral density of lift due to gust with $U\lambda = \nu$

$$= \frac{1}{U^2}\left|\left[\frac{1}{2\pi\lambda}\left(\frac{K_1(2\pi\lambda)}{K_0(2\pi\lambda) + K_1(2\pi\lambda)}\right)(J_0(2\pi\lambda) + i\,J_1(2\pi\lambda)) - 2i\,J_1(2\pi\lambda)\right]\right|^2 p(\lambda)$$

where we have used

$$x\,(J_0(x) + J_2(x)) = 2J_1(x)$$

10.3 CALCULATION OF HINGE MOMENT (M_α) DUE TO GUST

To calculate the hinge moment $M_{\alpha g}$ due to gust, we need to calculate:

$$M_{\alpha,g} = (-\rho)\left[U\int_{-1}^{1} x\,\gamma_g(t,x)dx - aU\Gamma_g(t) - \frac{d}{dt}\int_{-1}^{1} \frac{(x-a)^2}{2}\gamma_g(t,x)dx + \frac{(1-a)^2}{2}\Gamma_g'(t)\right]$$

Now,

$$\int_{-1}^{1} x\,\gamma_g(t,x)dx = \frac{-2}{\pi}\int_{-1}^{1} x\frac{\sqrt{1-x}}{\sqrt{1+x}}dx\int_{-1}^{1}\frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\frac{g(\zeta-Ut)}{x-\zeta}d\zeta + \int_0^t \left((1+U\sigma) - \sqrt{(1+U\sigma)^2-1}\right)\Gamma_g'(t-\sigma)d\sigma$$

$$= (+2)\int_{-1}^{1}\sqrt{(1-\zeta^2)}\,g(\zeta-Ut)d\zeta + \Gamma_g(t) - \Gamma_g(0) + \int_0^t c_3(\sigma)\,s(t-\sigma)d\sigma$$

While

$$\int_{-1}^{1}\frac{(x-a)^2}{2}\gamma_g(t,x)dx = \frac{+1}{\pi}\int_{-1}^{1}(x-a)^2\frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{1}{x-\zeta}dx\int_{-1}^{1}\frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}g(\zeta-Ut)d\zeta +$$

$$\int_0^t \frac{1}{2\pi^2}\int_{-1}^{1}(x-a)^2 H(\sigma,x)dx\;\Gamma_g'(t-\sigma)d\sigma$$

Now it can be shown that

$$\frac{1}{2} \cdot \frac{1}{\pi^2} \int_{-1}^{1} (x - a)^2 \, H(\sigma,x)dx$$

$$= \frac{1}{2} \left\{ \frac{1}{2} \frac{\sqrt{z + 1}}{\sqrt{z - 1}} - z^2 + z\sqrt{z^2 - 1} + 2a(z - \sqrt{z^2 - 1}) + a^2\left(\frac{\sqrt{z + 1}}{\sqrt{z - 1}} - 1\right) \right\} , \quad z = 1 + U\sigma$$

Now,

$$\frac{1}{\pi} \int_{-1}^{1} (x - a)^2 \frac{\sqrt{1 - x}}{\sqrt{1 + x}} \cdot \frac{1}{x - \zeta} \; dx = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - x}}{\sqrt{1 + x}} [x + (\zeta - 2a)]dx + \frac{(a - \zeta)^2}{x - \zeta} \; dx]$$

$$= -\frac{1}{2} + (\zeta - 2a) - (a - \zeta)^2$$

$$= -\frac{1}{2} - a^2 - \zeta^2 + 2a\zeta + \zeta - 2a$$

$$M_{\alpha,g}(t) = (-\rho) \left\{ (1 - a) \, \Gamma_g(t) - U \, \Gamma_g(0) \; + \; \frac{(1 - a)^2}{2} \Gamma_g{}'(t) + U\int_0^t c_3(\sigma) \; s(t - \sigma)d\sigma + 2U\int_{-1}^{1} \sqrt{1 - \zeta^2} \; g(\zeta - Ut)d\zeta \right.$$

$$\left. -\frac{1}{2} \frac{d}{dt} \int_0^t c_{12}(\sigma) \; \Gamma_g{}'(t - \sigma) \, d\sigma + \frac{d}{dt} \int_{-1}^{1} \frac{(\frac{1}{2} + a^2 + \zeta^2 - 2a\zeta - \zeta + 2a) \sqrt{(1 + \zeta)}}{\sqrt{1 - \zeta}} \; g(\zeta - Ut)d\zeta \right\}$$

From:

$$\Gamma_g{}'(t) = 2\pi \int_0^t c_1(t - \sigma)\, S(\sigma)d\sigma$$

we have that

$$\int_0^t c_{12}(\sigma)\, \Gamma_g{}'(t - \sigma)d\sigma = \int_0^t \{2a\, c_4(\sigma) + c_5(\sigma)\}\, S(t - \sigma)d\sigma + (\tfrac{1}{2} + a^2)\, S(t) - a^2\, \Gamma_g(t) ,$$

since (as can be verified by taking Laplace Transforms):

$$\int_0^t (\tfrac{1}{2} + a^2)\, \frac{\sqrt{2 + U\sigma}}{\sqrt{U\sigma}}\, c_1(t - \sigma)d\sigma = (\tfrac{1}{2} + a^2)\, \delta(t)$$

Hence,

$$M_{\alpha,g.}(t) = (-\rho)\{(1 - a)U\, \Gamma_g(t) - U\, \Gamma_g(0) + \frac{(1 - a)^2}{2}\, \Gamma_g{}'(t) + U \int_0^t c_3(\sigma)\, S(t - \sigma)d\sigma - U\, S_2(t)$$

$$- \tfrac{1}{2} \frac{d}{dt} \int_0^t (2a\, c_4(\sigma) + c_5(\sigma))\, S(t - \sigma)d\sigma + \frac{a^2}{2} \Gamma_g'(t) + \frac{d}{dt} \int_{-1}^1 (\zeta^2 - 2a\zeta - \zeta + 2a)\, \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}}\, g(\zeta - Ut)d\zeta\}$$

Now

$$\int_{-1}^{1} (\zeta^2 - 2a\zeta - \zeta + 2a) \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} e^{2\pi i\zeta\nu} \, d\zeta = \int_{-1}^{1} (2a) \sqrt{1-\zeta^2} \, e^{2\pi i\zeta\nu} \, d\zeta - \int_{-1}^{1} \zeta\sqrt{1-\zeta^2} \, e^{2\pi i\zeta\nu} \, d\zeta$$

$$= \pi a \, [J_0(2\pi\nu) + J_2(2\pi\nu)] - \frac{1}{2\pi i} \left(\frac{\pi}{2}\right) \frac{d}{d\nu} [J_0(2\pi\nu) + J_2(2\pi\nu)]$$

$$= \pi a \, [J_0(2\pi\nu) + J_2(2\pi\nu)] - \pi i \frac{J_2(2\pi\nu)}{2\pi\nu} = (\pi a) \frac{2J_1(2\pi\nu)}{2\pi\nu} - \pi i \frac{J_2(2\pi\nu)}{2\pi\nu}$$

where we have used:

$$J_2'(x) = -2 \frac{J_2(x)}{x} + J_1(x)$$

$$J_0'(x) = -J_1(x)$$

$$\frac{2J_1(x)}{x} = J_0(x) + J_1(x)$$

This shows that $M_{\alpha,g}(t)$ is asymptotically a stationary Gaussian process (if we may set $\Gamma_g(0) = 0$) with finite second moment. The spectral density can be deduced readily from the time-domain "convolution" integrals.

We have

$$M_\beta^g = (-\rho) \left\{ \int_c^1 U(x - c)\, \gamma_g(t,x)dx + \frac{d}{dt} \int_c^1 (x - c)dx \int_{-1}^x \gamma_g(t,y)dy \right\}$$

$$= (-\rho) \left\{ \int_c^1 U(x - c)\, \gamma_g(t,x)dx + \frac{(1 - c)^2}{2} \Gamma_g{}'(t) - \frac{d}{dt} \int_c^1 \frac{(x - c)^2}{2} \gamma_g(t,x)dx \right\}$$

$$\int_c^1 (x - c)\, \gamma_g(t,x)dx = (-1) \int_c^1 \left(\frac{2}{\pi}\right) (x - c) \frac{\sqrt{1 - x}}{\sqrt{1 + x}} dx \int_{-1}^1 \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} \frac{g(\zeta - Ut)}{x - \zeta} d\zeta$$

$$- \frac{1}{\pi^2} \int_0^t \int_c^1 (x - c) \frac{\sqrt{1 - x}}{\sqrt{1 + x}} H(\sigma,x)\, dx\, \Gamma_g{}'(t - \sigma)d\sigma = T_1 + T_2$$

The only way to handle the first term T_1 would appear to be to integrate with respect to x first, and then use the spectral representation for the gust. The second term T_2, of course, presents no problem. Thus, we have for the integral with respect to x in the first term:

$$(-1) \int_c^1 \left(\frac{2}{\pi}\right) (x - c) \frac{\sqrt{1 - x}}{\sqrt{1 + x}} \cdot \frac{1}{x - \zeta} dx = \frac{2}{\pi} \int_c^1 \left[x + (\zeta - 1 - c) + \frac{c + \zeta^2 - \zeta - c\zeta}{x - \zeta}\right] \cdot \frac{dx}{1 - x^2}$$

$$= \frac{2}{\pi} \left\{ \sqrt{1 - c^2} + (\zeta - 1 - c) \cos^{-1} c + \frac{(c + \zeta^2 - \zeta - c\zeta)}{\sqrt{1 - \zeta^2}} \cdot \log \left| \frac{\sqrt{(1 - c)(1 + \zeta)} + \sqrt{(1 + c)(1 - \zeta)}}{\sqrt{(1 - c)(1 + \zeta)} - \sqrt{(1 + c)(1 - \zeta)}} \right| \right\}$$

$$T_1 = \int_{-\infty}^{\infty} e^{-2\pi i \nu U t} g_1(\nu)\, dG(\nu)$$

where

$$g_1(\nu) = \frac{2}{\pi}\int_{-1}^{1} e^{2\pi i\nu\zeta}\left[(\sqrt{1-c^2} - (1+c)\cos^{-1}c + \frac{(1-\zeta)(c-\zeta)}{\sqrt{1-\zeta^2}}\log|\cdot|\right]\frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\,d\zeta$$

$$= \frac{2}{\pi}\int_{-1}^{1} e^{2\pi i\nu\zeta}\left[(\sqrt{1-c^2} - (1+c)\cos^{-1}c) + \cos^{-1}c\right]\frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\,d\zeta$$

$$+ \frac{2}{\pi}\int_{-1}^{1} e^{2\pi i\nu\zeta}\left[(c-\zeta)\log|\cdot|\right]d\zeta$$

Only the second integral requires attention. We use integration by parts to simplify it. Thus

$$\frac{d}{d\zeta}\log|\ | = \frac{\sqrt{1-c^2}}{\sqrt{1-\zeta^2}}\cdot\frac{1}{c-\zeta}$$

$$\int_{-1}^{1} e^{2\pi i\nu\zeta}(c-\zeta)d\zeta = \frac{c\, e^{2\pi i\nu\zeta}}{2\pi i\nu} + \frac{e^{2\pi i\times\zeta}}{(2\pi i\nu)^2} - \frac{e^{2\pi i\nu\zeta}}{(2\pi i\nu)}\zeta$$

$$= \frac{e^{2\pi i\nu\zeta}}{(2\pi i\nu)}\left[(c-\zeta) + \frac{1}{2\pi i\nu}\right]$$

$$
\begin{aligned}
\frac{2}{\pi}\int_{-1}^{1} e^{2\pi i\nu\zeta}(c-\zeta)\log|\cdot|\,d\zeta &= \frac{2}{\pi}\int_{-1}^{1}\frac{e^{2\pi i\nu\zeta}}{2\pi i\nu}\left[(c-\zeta)+\frac{1}{2\pi i\nu}\right]\cdot\frac{\sqrt{1-c^2}}{\sqrt{1-\zeta^2}}\cdot\frac{1}{(c-\zeta)}\,d\zeta \\
&= \frac{2}{\pi}\int_{-1}^{1}\left(\frac{1}{2\pi i\nu}\right)^2\frac{\sqrt{1-c^2}}{\sqrt{1-\zeta^2}}\,\frac{e^{2\pi i\nu\zeta}}{c-\zeta}\,d\zeta \\
&= \left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi i\nu}\right)^2\sqrt{1-c^2}\int_{-1}^{1}\frac{1}{\sqrt{1-\zeta^2}}\cdot\frac{1}{c-\zeta}\,e^{2\pi i\nu\zeta}\,d\zeta \\
&= \left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi i\nu}\right)^2\sqrt{1-c^2}\,f(c;\nu)
\end{aligned}
$$

where we define

$$
f(c;\nu) = \int_{-1}^{1}\frac{1}{\sqrt{1-\zeta^2}}\;\frac{1}{c-\zeta}\,e^{2\pi i\nu\zeta}\,d\zeta
$$

Then

$$
\begin{aligned}
g_1(\nu) = \left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi i\nu}\right)^2\sqrt{1-c^2}\,f(c;\nu) &+ (2)\left(\sqrt{1-c^2}-(1+c)\cos^{-1}c\right)\left(J_0(2\pi\nu)+i\,J_1(2\pi\nu)\right) \\
&+ 2(\cos^{-1}c)\left[J_0(2\pi\nu)-\frac{J_1(2\pi\nu)}{2\pi\nu}+i\,J_1(2\pi\nu)\right]
\end{aligned}
$$

Next in T_2:

$$\frac{1}{\pi^2} \int_0^t \int_c^1 (x - c) \frac{\sqrt{1 - x}}{\sqrt{1 + x}} H(\sigma,x)dx\, \Gamma_g'(t - \sigma)d\sigma$$

$$= \frac{1}{\pi^2} \int_0^t H_1(\sigma)\, \Gamma_g'(t - \sigma)d$$

where

$$H_1(\sigma) = (-4)\ (z - c) \tan^{-1} \frac{\sqrt{(z + 1)(1 - c)}}{\sqrt{(z - 1)(1 + c)}} - 2 \frac{\sqrt{z + 1}}{\sqrt{z - 1}} [(1 - z + c) \cos^{-1} c - \sqrt{1 - c^2}$$

Next we need to calculate:

$$\frac{d}{dt} \int_c^1 \frac{(x - c)^2}{2}\ \gamma_g(t,x)dx = \frac{d}{dt} T_3 + \frac{d}{dt} T_4$$

where

$$T_3 = \left(\frac{-2}{\pi}\right) \int_c^1 \frac{(x - c)^2}{2} \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\, dx \int_{-1}^1 \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} \frac{g(\zeta - Ut)}{x - \zeta}\, d\zeta$$

Now

$$\int_c^1 (x - c)^2 \frac{\sqrt{1 - x}}{\sqrt{1 + x}} \cdot \frac{1}{x - \zeta} \, dx$$

$$= \int_c^1 [-x^2 + (1 + 2c + \zeta)x + (\zeta + 2c \;\; + \zeta^2 - c^2 - 2c)] \frac{dx}{\sqrt{1 - x^2}} + \int_c^1 \frac{(\zeta - c)^2 (1 - \zeta)}{\sqrt{1 - x^2}} \frac{dx}{x - \zeta}$$

$$= m_1(\zeta) + \frac{(\zeta - c)^2 (1 - \zeta)}{\sqrt{1 - \zeta^2}} \log \left| \frac{\sqrt{(1 - c)(1 + \zeta)} + \sqrt{(1 + c)(1 - \zeta)}}{\sqrt{(1 - c)(1 + \zeta)} - \sqrt{(1 + c)(1 - \zeta)}} \right|$$

Hence

$$T_3 = -\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} m_1(\zeta) \, g(\zeta - Ut) d\zeta - \frac{1}{\pi} \int_{-1}^1 (\zeta - c)^2 \, g(\zeta - Ut) \log | \; | \, d\zeta$$

Now let

$$\int_{-1}^1 (\zeta - c)^2 \, e^{2\pi i \nu \zeta} \log | \; | \, d\zeta = e^{2\pi i \nu c} \int_{-1}^1 (\zeta - c)^2 \, e^{2\pi i \nu (\zeta - c)} \log | \; | \, d\zeta$$

$$= \frac{e^{2\pi i \nu c}}{(2\pi i)^2} \frac{d^2}{d\nu^2} \int_{-1}^1 e^{2\pi i \nu (\zeta - c)} \log | \; | \, d\zeta$$

Hence it is enough to calculate

$$\int_{-1}^{1} e^{2\pi i\nu\zeta} \log |\ | \, d\zeta = \frac{f(c;\nu)}{2\pi i\nu}$$

Hence, finally,

$$T_3 = -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2\pi i\nu Ut} [h_3(\nu)] \, dG(\nu)$$

where

$$h_3(\nu) = \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} m_1(\zeta) \, e^{2\pi i\nu\zeta} \, d\zeta + \frac{e^{2\pi i\nu c}}{(2\pi i)^2} \frac{d^2}{d\nu^2} \left[e^{-2\pi i\nu c} \frac{f(c,\nu)}{2\pi i\nu} \right]$$

Next

$$\frac{d}{dt} T_4 = \frac{-1}{2} \frac{1}{2\pi} dt \int_0^t H_2(\sigma) \, \Gamma_g{}'(t-\sigma) d\sigma$$

where

$$H_2(\sigma) = \frac{2}{\pi} \int_c^1 (x - c)^2 \frac{\sqrt{1 - x}}{\sqrt{1 + x}} H(\sigma,x)dx$$

$$= (-4)(z - c)^2 \tan^{-1} \frac{\sqrt{(1 + z)(1 - c)}}{\sqrt{(z - 1)(1 + c)}} + \frac{\sqrt{z + 1}}{\sqrt{z - 1}} \left[(1 - 2z + 4c + 2(z - c)^2)\cos^{-1}c - (2 + 3c - 2z)\sqrt{1 - c^2} \right]$$

Hence, finally,

$$\begin{aligned} M_\beta^g(t) = (-\rho) \Bigg\{ & \frac{2U}{\pi} \int_{-1}^{1} \frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} \Bigg[\sqrt{1 - c^2} + (\zeta - 1 - c)\cos^{-1}c + \\ & + \frac{(c - \zeta)(1 - \zeta)}{\sqrt{1 - \zeta^2}} \log \left| \frac{\sqrt{(1 - c)(1 + \zeta)} + \sqrt{(1 + c)(1 - \zeta)}}{\sqrt{(1 - c)(1 + \zeta)} - \sqrt{(1 + c)(1 - \zeta)}} \right| \Bigg] g(\zeta - Ut)d\zeta - \frac{U}{2\pi} \int_0^t H_1(\sigma)\, \Gamma_g'(t - \sigma)d\sigma \\ & + \frac{(1 - c)^2}{2} \Gamma_g'(t)\ (+\frac{1}{\pi}) \frac{d}{dt} \int_{-1}^{1} \left[\frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}} m_1(\zeta) + (\zeta - c)^2 \log |\ | \right] g(\zeta - Ut)d\zeta \\ & + \frac{1}{4} \frac{d}{dt} \int_0^t H_2(\sigma)\, \Gamma_g'(t - \sigma)d\sigma \Bigg\} \end{aligned}$$

which is the time-domain version of the flap moment due to gust. It is seen to be Gaussian, and asymptotically stationary, with spectral density that can be readily deduced from:

$$M_\beta^g(\tau) = (-\rho)\Bigg\{ U \int_{-\infty}^{\infty} \Bigg\{ \left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi i\nu}\right)\sqrt{1 - c^2}\, f(c;\nu) + 2(\sqrt{1 - c^2} - (1 + c)\cos^{-1}c)(J_0(2\pi\nu) + i\, J_1(2\pi\nu))$$

$$+ 2(\cos^{-1}c)\left[J_0(2\pi\nu) - \frac{(J_1(2\pi\nu) - 2\pi i\nu\, J_1(2\pi\nu))}{2\pi\nu}\right]\Bigg\} e^{-2\pi i\nu Ut}\, dG(\nu) - \frac{U}{2\pi}\int_0^t H_1(\sigma)\, \Gamma_g{}'(t - \sigma)d\sigma$$

$$+ \frac{(1 - c)^2}{2}\Gamma_g{}'(t) + \frac{1}{4\pi}\frac{d}{dt}\int_0^t H_2(\sigma)\, \Gamma_g{}'(t - \sigma)d\sigma + \frac{1}{\pi}\int_{-\infty}^{\infty} h_3(\nu)\, e^{-2\pi i\nu Ut}\, dG(\nu)\Bigg\}$$

where

$$h_3(\nu) = \frac{e^{2\pi i\nu c}}{(2\pi i)^2}\,\frac{d^2}{d^{\ 2}}\left[e^{-2\pi i\nu c}\ \frac{f(c,\nu)}{2\pi i\nu}\right]$$

$$+ \int_{-1}^{1} e^{2\pi i\nu\zeta}\,\frac{\sqrt{1 + \zeta}}{\sqrt{1 - \zeta}}\left[(\zeta + 2c\zeta + \zeta^2 - c^2 - 2c - \tfrac{1}{2})\cos^{-1}c + \sqrt{1 - c^2}\,(1 + \frac{3c}{2} + \zeta)\right] d\zeta$$

10.5 CALCULATION OF STRUCTURAL RESPONSE TO GUST

The structural response is governed by the dynamic equations:

$$M_s \ddot{x} + B_s \dot{x} + K_s x = \left(\frac{1}{m_s}\right) \begin{bmatrix} P_g(t) \\ M_\alpha^g(t) \\ M_\beta^g(t) \end{bmatrix}$$

where $P_g(t)$, $M_\alpha^g(t)$, $M_\beta^g(t)$ have already been calculated. From these calculations, we can see that the right-hand side can be expressed as:

$$\int_{-\infty}^{\infty} e^{2\pi i \nu t} L(\nu)\, dG(\nu)$$

where, however, the transfer function $L(\nu)$ is not necessarily "physically realizable". In other words, we cannot express it as:

$$L(\nu) = \int_0^{\infty} e^{-2\pi i \nu t} W(t) dt$$

where $W(\cdot)$ would be the "weighting matrix" corresponding to a physical system (vanishing for negative t-values). The gust itself can, if we follow the Dryden form of the spectral density, be expressed as:

$$g(x) = \int_{-\infty}^{\infty} e^{2\pi i \nu x} \; h(\nu) \; dN(\nu)$$

where

$$h(\nu) = \frac{\text{const}}{1 + i\,k_2 \nu}$$

and

$$E(|dN(\nu)|^2) = d\nu$$

and $h(\nu)$ is of course a physically realizable transfer function.

It would be interesting to approximate $L(\nu)$ by a rational function.

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INTEGRAL FORMULAS

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Formulas

1. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \, d\zeta = \pi; \qquad \int_{-1}^{1} \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{x-\zeta} \, d\zeta = 0 \quad |x| < 1 \qquad \text{(cf. 6a)}$$

2. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{d\zeta}{x-\zeta} = \int_{-1}^{1} \frac{1}{\sqrt{1-\zeta^2}} \frac{1+\zeta}{x-\zeta} \, d\zeta$$

$$= \int_{-1}^{1} \frac{(-1)}{\sqrt{1-\zeta^2}} \left[1 - \frac{(x+1)}{x-\zeta}\right] d\zeta = -\pi, \ |x| < 1$$

3. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\zeta}{x-\zeta} \, d\zeta = \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \left[\frac{x}{x-\zeta} - 1\right] d\zeta = = \pi x - \pi$$

4. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z} \, d\zeta =$$

Now $$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z} \, d\zeta = \int_{c}^{1} \frac{1}{\sqrt{1-\zeta^2}} \left[\frac{1+\zeta}{\zeta-z}\right] d\zeta$$

$\cos\phi = c$ $$= \int_{c}^{1} \frac{1}{\sqrt{1-\zeta^2}} \left[1 + \frac{z+1}{\zeta-z}\right] d\zeta$$

$$= \phi + (z+1) \int_{c}^{1} \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{\zeta-z} \, d\zeta$$

$$\int_c^1 \frac{1}{\sqrt{1-\zeta^2}} \qquad \frac{1}{\zeta-z}\, d\zeta$$

$$\zeta = \frac{1-t^2}{1+t^2} \qquad 1 - \zeta^2 = 1 - \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2}{(1+t^2)^2}$$

$$\sqrt{1-\zeta^2} = \frac{2t}{(1+t^2)}$$

$$t^2 = \frac{1-\zeta}{1+\zeta}$$

$$d\zeta = \frac{((1+t^2)(-2t) - (1-t^2)(2t))dt}{(1+t^2)^2} = -\frac{4t\, dt}{(1+t^2)^2}$$

$$\int_c^1 \frac{1}{\sqrt{1-\zeta^2}} \quad \frac{1}{\zeta-z}\, d\zeta = \int_0^{\sqrt{\frac{1-c}{1+c}}} \frac{(1+t^2)}{2t} \quad \frac{(1+t^2)}{(1-t^2)-z(1+t^2)} \qquad \frac{4t}{(1+t^2)^2}\; dt$$

$$= (-1) \int_0^{\sqrt{\frac{1-c}{1+c}}} (2) \quad \frac{dt}{(z-1)+t^2(1+z)}$$

$$\text{If } z > 1 \quad \Rightarrow \quad = (-2) \int_0^{\sqrt{\frac{1-c}{1+c}}} \frac{dt}{t^2(1+z)+(z-1)}$$

$$= \frac{(-2)}{1+z} \left[\frac{\sqrt{z+1}}{\sqrt{z-1}} \operatorname{Tan}^{-1} \frac{t\sqrt{z+1}}{\sqrt{z-1}} \right]_0^{\sqrt{(1-c)/1+c}}$$

5. $$\int_c^1 \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{\zeta-z}\, d\zeta = \frac{(-2)}{\sqrt{z^2-1}} \operatorname{Tan}^{-1} \frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}} \quad , \quad z > 1$$

If $z < 1$, $$\int_c^1 \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{\zeta-z}\, d\zeta = \frac{(-2)}{1+z} \int_0^{\sqrt{\frac{1-c}{1+c}}} \frac{dt}{t^2 - \frac{(1-z)}{1+z}}$$

$$= \frac{-2}{2\sqrt{1-z^2}} \left[\operatorname{Log} \left| \frac{t-\sqrt{1-z}/\sqrt{1+z}}{t+\sqrt{1-z}/\sqrt{1+z}} \right| \right]_0^{\sqrt{\frac{1+c}{1-c}}}$$

6. $$\int_c^1 \frac{1}{\sqrt{1-\zeta^2}} \frac{d\zeta}{\zeta-z} = + \frac{1}{\sqrt{1-z^2}} \operatorname{Log} \left| \frac{\sqrt{(1-c)}\sqrt{1+z} + \sqrt{(1+c)}\sqrt{1-z}}{\sqrt{(1-c)}\sqrt{1+z} - \sqrt{(1+c)}\sqrt{1-z}} \right| \quad , \quad |z| < 1$$

5(a). $$\int_{-1}^1 \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{\zeta-z}\, d\zeta = \frac{-\pi}{\sqrt{z^2-1}} \quad , \quad z > 1$$

6(a). $$\int_{-1}^{1} \frac{1}{\sqrt{1-\zeta^2}} \frac{1}{\zeta-z}\, d\zeta = 0 \quad z < 1$$

Hence

7. $$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z}\, d\zeta = \cos^{-1}c + \sqrt{\frac{1+z}{1-z}}\ \mathrm{Log} \left| \frac{\sqrt{(1-c)}\sqrt{1+z} + \sqrt{(1+c)}\sqrt{1-z}}{\sqrt{(1-c)}\sqrt{1+z} - \sqrt{(1+c)}\sqrt{1-z}} \right| , \quad z^2 < 1$$

8. $$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z}\, d\zeta = \cos^{-1}c - 2\sqrt{\frac{z+1}{z-1}}\ \mathrm{Tan}^{-1} \frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}} , \quad z > 1$$

9. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z}\, d\zeta = \pi \quad z < 1$$

10. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-z}\, d\zeta = \pi - \pi \frac{\sqrt{z+1}}{\sqrt{z-1}} ; \quad z > 1$$

10a. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{(\zeta-a)}{x-\zeta}\, d\zeta = \pi a - \pi - \pi x, \quad |x| < 1$$

10b. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{(\zeta-c)}{x-\zeta}\, d\zeta = (-\cos^{-1}c - \sqrt{1-c^2}) - [\cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log |\ |] (x-c)$$

11. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{\zeta-1-i\sigma} \frac{1}{x-\zeta} d\zeta = \int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-z} \left(\frac{1}{\zeta-1-i\sigma} + \frac{1}{x-\zeta}\right) d\zeta, \quad z = 1 + i\sigma$$

$$= -\frac{\pi}{x-z} + \frac{\pi}{x-z} - \frac{\pi}{x-z} \frac{\sqrt{z+1}}{\sqrt{z-1}}$$

$$= -\frac{\pi}{(x-z)} \frac{\sqrt{z+1}}{\sqrt{z-1}}$$

12. $$H(\sigma) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} H(\sigma,x)dx$$

$$= \left(\frac{2}{\pi}\right)(-\pi) \frac{\sqrt{z+1}}{\sqrt{z-1}} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{1}{x-z} dx$$

$$= \left(\frac{2}{\pi}\right)(-\pi) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(\frac{1-z}{x-z} - 1\right)dx\right)$$

$$= (-2) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left(\frac{(1-z)\ (-2)}{\sqrt{z^2-1}} \frac{\pi}{2} - \pi\right)$$

$$= (2\pi) \frac{\sqrt{z+1}}{\sqrt{z-1}} \left(1 + \frac{1-z}{\sqrt{z^2-1}}\right)$$

$$= (2\pi) \left(\frac{\sqrt{z+1}}{\sqrt{z-1}} - 1\right)$$

13. $$\int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{1}{x-\zeta}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(\frac{1-\zeta}{x-\zeta} - 1\right) dx = -\pi, \quad \zeta < 1$$

13(a). $$\int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{1}{x-z}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(\frac{1-z}{x-z} - 1\right) dx = \frac{\pi(z-1)}{\sqrt{z^2-1}} - \pi, \quad z > 1$$

14. $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \zeta\, d\zeta = \int_{-\pi/2}^{\pi/2} (1+\sin\theta)\sin\theta\, d\theta$$

$$= \frac{\pi}{2}$$

15. $$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}}\, d\zeta = \int_{\phi}^{\pi/2} (1+\sin\theta)\, d\theta = \left(\frac{\pi}{2} - \phi\right) + \sqrt{1-c^2}$$

$$= \cos^{-1} c + \sqrt{1 - c^2}$$

$$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \zeta\, d\zeta = \int_{\phi}^{\pi/2} (1+\sin\theta)\sin\theta\, d\theta = \sqrt{1-c^2} + \frac{1}{2} \cos^{-1} c + \frac{1}{2} c\sqrt{1-c^2}$$

$$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (\zeta-c)\, d\zeta = \sqrt{1-c^2} + \left(\frac{1}{2} - c\right) \cos^{-1} c = \left(1 - \frac{c}{2}\right) \sqrt{1-c^2} + \left(\frac{1}{2} - c\right)\cos^{-1} c$$

16. $$\int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \zeta^2 d\zeta = \int_\phi^{\pi/2} (1+\sin\theta)\sin^2\theta d\theta$$

$\sin\phi = c$

$$= \int_\phi^{\pi/2} \sin^2\theta d\theta + \int_\phi^{\pi/2} \sin^3\theta d\theta$$

$$= \frac{1}{2} \cos^{-1}c + \frac{1}{2} c\sqrt{1-c^2} + \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_\phi^{\pi/2}$$

17. $$\int_{-1}^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} x\, dx = \int_{-1}^1 \frac{(1-x)x}{\sqrt{1-x^2}} dx = (-1) \int_{-\pi/2}^{\pi/2} \sin^2\theta d\theta$$

$$= -\frac{\pi}{2}$$

18. $$\int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (-1+2\zeta-2\zeta^2)d\zeta$$

$$= \int_\phi^{\pi/2} (1+\sin\theta)(-1+2\sin\theta-2\sin^2\theta)d\theta$$

$$= \int_\phi^{\pi/2} (-1+2\sin\theta-2\sin^2\theta-\sin\theta+2\sin^2\theta-2\sin^3\theta)d\theta$$

$$= \int_{\phi}^{\pi/2} (-1+\sin\theta-2\sin^3\theta)\,d\theta$$

$$= -\cos^{-1}c - \sqrt{1-c^2} + \frac{2}{3}\left(\sqrt{1-c^2}\right)^3$$

19. $$\int_{c}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (-1+2\zeta-2\zeta^2)\zeta\,d\zeta$$

$$= \int_{\phi}^{\pi/2} (-\sin\theta+\sin^2\theta-2\sin^4\theta)\,d\theta$$

$$= (-1)\sqrt{1-c^2} + \frac{1}{2}\cos^{-1}c + \frac{1}{2}c\sqrt{1-c^2} - 2\left[\frac{3\theta}{8} - \frac{3\sin 2\theta}{16} - \frac{\sin^3\theta\cos\theta}{4}\right]_{\phi}^{\pi/2}$$

$$= \left(-1 - \frac{c}{4} - \frac{c^3}{4}\right)\sqrt{1-c^2} - \frac{1}{4}\cos^{-1}c$$

18(a) $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (-1+2\zeta-2\zeta^2)\,d\zeta = -\pi$$

19(a) $$\int_{-1}^{1} \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} (-1+2\zeta-2\zeta^2)\zeta\,d\zeta = -\frac{\pi}{4}$$

20. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}}(x-c)dx = \int_\theta^{\pi/2} (1-\sin\theta)(\sin\theta-c)d\theta$$

$$= (1+c)\sqrt{1-c^2} - c\cos^{-1}c - \frac{1}{2}\cos^{-1}c - \frac{1}{2}c\sqrt{1-c^2}$$

$$= (1+\frac{c}{2})\sqrt{1-c^2} - (c+\frac{1}{2})\cos^{-1}c.$$

21. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}}x(x-c)dx = \int_\phi^{\pi/2} ((1+c)\sin^2\theta - c\sin\theta - \sin^3\theta)d\theta$$

$$= \int_\phi^{\pi/2} ((\frac{1+c}{2})\frac{(1-\cos^2\theta)}{2} - c\sin\theta - \sin^3\theta)d\theta$$

$$= [\,(\frac{1+c}{2})\theta - \frac{(1+c)}{4}\sin^2\theta + c\cos\theta - \frac{\cos^3\theta}{3} + \cos\theta]_\phi^{\pi/2}$$

$$= \frac{1}{2}(1+c)\cos^{-1}c + \frac{(1+c)}{2}c\sqrt{1-c^2} - c\sqrt{1-c^2} + \frac{(\sqrt{1-c^2})^3}{3} - \sqrt{1-c^2}$$

$$= \frac{1}{2}(1+c)\cos^{-1}c + \frac{1}{2}\sqrt{1-c^2}(c^2-c-2) + \frac{(\sqrt{1-c^2})^3}{3}$$

22. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-1)(x-c)\,dx = -\frac{c}{2} \cos^{-1} c + \frac{c^2}{2} (\sqrt{1-c^2}) + \frac{(\sqrt{1-c^2})^3}{3}$$

23. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)^2\,dx = \int_\phi^{\pi/2} (\sin\theta - c)^2 (1-\sin\theta)\,d\theta$$

$$= \Big[_\phi^{\pi/2} (c^2 + \frac{1}{2} - c)\theta - \frac{(1-2c)}{4} \sin 2\theta + (1+c)^2 \cos\theta - \frac{\cos^3\theta}{3} \Big]$$

$$= \Big[\frac{1}{2} (1+2c+2c^2) \cos^{-1} c + \frac{(1-2c)c\sqrt{1-c^2}}{2} - (1+c)^2\sqrt{1-c^2} + \frac{(\sqrt{1-c^2})^3}{3} \Big]$$

$$= \Big[\frac{(1+2c+2c^2)}{2} \cos^{-1} c + \frac{1}{6} (-2c^2-9c-4)\sqrt{1-c^2}\Big]$$

$$= h_1(c)$$

24. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x+1)(x-c)^2\,dx = \int_\phi^{\pi/2} (\sin\theta - c)^2(-\sin^2\theta + 1)\,d\theta$$

$$= h_2(c)$$

25. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx = \int_\phi^{\pi/2} (1-\sin\theta)\, d\theta = \cos^{-1} c - \sqrt{1-c^2}$$

26. $$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} \frac{1}{x-z}\, dx = \int_c^1 \frac{1}{\sqrt{1-x^2}} \left[-1 + \frac{1-z}{x-z}\right] dx$$

$$= -\cos^{-1} c + \frac{2(z-1)}{\sqrt{z^2-1}} \operatorname{Tan}^{-1} \frac{(1-c)\sqrt{z+1}}{(1+c)\sqrt{z-1}}, \quad z > 1.$$

27. $$\int_c^1 x \frac{\sqrt{1-x}}{\sqrt{1+x}}\, dx = \int_\phi^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta = \sqrt{1-c^2} - \frac{1}{2} \cos^{-1} c - \frac{c\sqrt{1-c^2}}{2}$$

28. $$f_1(t) = \frac{2}{\pi} \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)\, dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta}\, d\zeta = +\frac{1}{\pi}\left[(1+2c)(\cos^{-1} c)^2 - 2\sqrt{1-c^2}\cos^{-1} c - (1-c^2)\right]$$

29. $$f_2(c) = \frac{2}{\pi} \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)\, dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{\zeta}{x-\zeta}\, d\zeta \rightarrow = \left(-\frac{1}{\pi}\right) \left\{ (\cos^{-1} c + \sqrt{1-c^2})\left[(1+c)\cos^{-1} c + (c^2 - c - 2)\sqrt{1-c^2} + \frac{2}{3}(\sqrt{1-c^2})^3\right] + \frac{1}{3} (\sqrt{1-c^2})^3 \cos^{-1} c\right\}$$

let

$$\psi(t) = \frac{1}{2\pi} \int e^{st} \frac{s^{-1}e^{-s}}{K_0(s) + K_1(s)} ds$$

Then

$$c_1(t) = \frac{1}{2\pi} \int e^{st} s^{-1} \frac{e^{-s/U} U}{K_0(s/U + K_1(s/U)} ds$$

$$= \frac{1}{2\pi} \int e^{s(Ut)} U \frac{s^{-1}e^{-s}}{K_0(s) + K_1(s)} ds$$

$$= U\psi(Ut)$$

$$c(t) = U^2\psi'(Ut)$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-a)^2 dx = (\frac{1}{2} + a^2 + a); \quad \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} \; x \; (x-a)^2 dx = - a - \frac{a^2}{2} - \frac{3}{8}$$

$$\frac{1}{\pi} \int_{-1}^{1} (x-a)^2 (x-c) \log \; |\cdot| \; dx = \frac{1}{12} \sqrt{1-c^2} \; [- \frac{(8a+c)}{2} + 4ac^2 - 6a^2 c - c^3]$$

$$\frac{1}{\pi} \int_{-1}^{1} (x-a)^2 \log \; |\cdot| \; dx = \frac{1}{\pi} \int_{-1}^{1} \frac{(x-a)^3}{3} \cdot \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{dx}{x-c} = \frac{1}{6} + 2a^2 + \frac{2c^2}{3} - 2ac$$

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A P P E N D I X T W O

SPECIAL FUNCTIONS

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Calculation of functions $f_1(c)$ and $f_2(c)$, $g_1(c)$ and $g_2(c)$.

$$f_1(c) = \frac{2}{\pi} \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} d\zeta$$

$$= \frac{2}{\pi} \left[- \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)dx \cos^{-1}c - \int_c^1 (x-c) \log \frac{\sqrt{(1-c)(1+x)} + \sqrt{(1+c)(1-x)}}{\left|\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}\right|} dx\right]$$

where we have used:

$$\int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} d\zeta = - \cos^{-1} c - \frac{\sqrt{1+x}}{\sqrt{1-x}} \log \frac{\sqrt{(1-c)(1-x)} + \sqrt{(1+c)(1-x)}}{\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}}$$

Now the second integral in brackets can be integrated by parts noting

$$\frac{d}{dx} \log \frac{\sqrt{(1-c)(1+x)} + \sqrt{(1+c)(1-x)}}{\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}}$$

$$= - \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{1}{x-c}$$

We have:

$$\int_c^1 (x-c)\log \frac{\sqrt{(1-c)(1+x)} + \sqrt{(1+c)(1-x)}}{\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}} dx$$

$$= \int_c^1 \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{(x-c)^2}{2} \cdot \frac{1}{x-c} dx = \frac{1}{2} \sqrt{1-c^2} \int_0^\phi (\cos\theta - c)d\theta$$

$$= \frac{1}{2} (\sqrt{1-c^2}) [\sqrt{1-c^2} - c \cos^{-1}c]$$

Now the first integral:

$$\int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} \ (x-c)dx$$

$$= \frac{1}{2} \ [(2+c)\sqrt{1-c^2} \ - (2c+1) \ \cos^{-1}c]$$

Hence

$$f_1(c) = \frac{1}{\pi} \ [(2c+1)(\cos^{-1}c)^2 - (2+c)\sqrt{1-c^2} \ \cos^{-1}c - (1-c^2) + c\sqrt{1-c^2} \ \cos^{-1}c]$$

$$= \frac{1}{\pi} \ [(2c+1)(\cos^{-1}c)^2 - (1-c^2) - 2\sqrt{1-c^2} \ \cos^{-1}c]$$

Next

$$f_2(c) = \frac{2}{\pi} \ \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)dx \ \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \cdot \frac{\zeta}{x-\zeta} \ d\zeta$$

$$= (\frac{-2}{\pi}) \ \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \ (1- \frac{x}{x-\zeta}) \ d\zeta$$

$$= (\frac{-2}{\pi}) \ \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x-c)dx \ \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} d\zeta$$

$$+ (\frac{-2}{\pi}) \ \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} \ (x^2-cx)dx \ \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \ \frac{d\zeta}{x-\zeta}$$

The first term is

$$= (\frac{-1}{\pi}) \ [(1+c)\cos^{-1}c + (\sqrt{1-c^2})(c^2-c-2) + \frac{2}{3} (\sqrt{1-c^2})^3] \cdot (\cos^{-1}c + \sqrt{1-c^2}]$$

The second term is:

$$- \frac{2}{\pi} \int_c^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} (x^2 - cx) \cdot \frac{\sqrt{1+x}}{\sqrt{1-x}} \log | \quad | \, dx$$

and by integration by parts

$$= - \frac{2}{\pi} \int_c^1 (\frac{x^3}{3} - \frac{cx^2}{2}) \cdot \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{1}{x-c} \, dx$$

$$= - \frac{1}{3\pi} \int_c^1 \frac{(\sqrt{1-c^2})}{\sqrt{1-x^2}} [2x^2 - cx - c^2 - \frac{c^3}{x-c}] dx$$

$$= - \frac{1}{3\pi} \int_0^{\phi} \sqrt{1-c^2} \, (2 \cos^2\theta - c \cos\theta - c^2] d\theta$$

since a ready integration shows that

$$\int_c^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{x-c} \, dx = 0$$

Hence 2nd term

$$= - \frac{1}{3\pi} \sqrt{1-c^2} \, [\cos^{-1}c + c\sqrt{1-c^2} - c\sqrt{1-c^2} - c^2 \cos^{-1}c]$$

$$= - \frac{1}{3\pi} \sqrt{1-c^2} \, (1-c^2) \cos^{-1}c$$

Hence

$$f_2(c) = \frac{(-1)}{\pi} \{[(1+c)\cos^{-1}c + (c^2-c-2)\sqrt{1-c^2} + \frac{2}{3}(\sqrt{1-c^2})^3](\cos^{-1}c + \sqrt{1-c^2}) + \frac{1}{3}(\sqrt{1-c^2})^3 \cos^{-1}c$$

$\underline{g_1(c)}$

$$g_1(c) = \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \frac{1}{x-\zeta} d\zeta$$

$$= \frac{2}{\pi} \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \left[- \cos^{-1}c - \frac{\sqrt{1+x}}{\sqrt{1-x}} \log | \quad |\right] dx$$

$$= \frac{2}{\pi} (- \cos^{-1}c) \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx$$

$$- \frac{2}{\pi} \int_c^1 (x-c)^2 \log | \quad | dx$$

2nd term, by integration by parts

$$= \frac{-2}{\pi} \int_c^1 \frac{(x-c)^3}{3} \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{1}{x-c} dx$$

$$= - \frac{2}{3\pi} \int_c^1 \sqrt{1-c^2} \frac{(x-c)^2}{\sqrt{1-x^2}} dx$$

$$= - \frac{2}{3\pi} \int_0^\phi \sqrt{1-c^2} (\cos^2\theta + c^2 - 2c \cos\theta) d\theta$$

$$: = - \frac{2\sqrt{1-c^2}}{3\pi} \left[\frac{\cos^{-1}c}{2} + \frac{c\sqrt{1-c^2}}{2} + c^2\cos^{-1}c - 2c\sqrt{1-c^2}\right]$$

$$= - \frac{1}{3\pi} \sqrt{1-c^2} \left[(1+2c^2) \cos^{-1}c - \frac{3}{2}c\sqrt{1-c^2}\right]$$

The first term

$$= -\frac{2}{\pi}\cos^{-1}c \qquad \int_c^1 (x-c)^2 \ \frac{\sqrt{1-x}}{\sqrt{1+x}} \ dx$$

$$= -\frac{2}{\pi}\cos^{-1}c \left[\left(\frac{1-2c+2c^2}{2}\right) \ \cos^{-1}c - \frac{1}{6} \ (2c^2 + 9c + 4)\sqrt{1-c^2}\ \right]$$

Hence

$$g_1(c) = (\sqrt{1-c^2})\left[\frac{c\sqrt{1-c^2}}{2\pi} - \frac{(1+2c^2)}{3\pi}\cos^{-1}c + \frac{1}{3\pi}(2c^2 + 9c + 4)\right]$$

$$- \frac{1}{\pi}(\cos^{-1}c)^2 \ (1-2c+2c^2)$$

<u>$g_2(c)$</u>

$$g_2(c) = \frac{2}{\pi} \quad \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \quad \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \ \frac{\zeta}{x-\zeta} \ d\zeta$$

$$= \frac{(-2)}{\pi} \quad \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \left[\int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \ \left(1 + \frac{x}{\zeta-x}\right)d\zeta\right]$$

$$= -\frac{2}{\pi} \quad \int_c^1 (x-c)^2 \ \frac{\sqrt{1-x}}{\sqrt{1+x}} \ dx \cdot (\cos^{-1}c + \sqrt{1-c^2})$$

$$= -\frac{2}{\pi} \quad \int_c^1 (x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} \ x \cdot \quad \int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \cdot \frac{1}{\zeta-x} \ d\zeta$$

Now

$$\int_c^1 \frac{\sqrt{1+\zeta}}{\sqrt{1-\zeta}} \cdot \frac{1}{\zeta-x} \, d\zeta$$

$$= \cos^{-1}c + \frac{\sqrt{1+x}}{\sqrt{1-x}} \log |\quad|$$

Hence 2nd term in $g_2(c)$ is:

$$- \frac{2}{\pi} \int_c^1 x(x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \cos^{-1}c$$

$$- \frac{2}{\pi} \int_c^1 x(x-c)^2 \log || \, dx$$

$$= - h_3(c) - \frac{2}{\pi} \int_c^1 \left(\frac{x^4}{4} + \frac{c^2x^2}{2} - \frac{2cx^3}{3} - \frac{c^4}{12}\right) \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \cdot \frac{dx}{x-c}$$

Now

$$h_3(c) = \frac{2}{\pi} \int_c^1 x(x-c)^2 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx$$

$$= \frac{2}{\pi} \int_0^\phi (\cos\theta - c)^2 \cos\theta \, (1 - \cos\theta) \, d\theta$$

$$= \frac{2}{\pi} \int_0^\phi (-\cos^4\theta + (1+2c)\cos^3\theta - (c^2 + 2c)\cos^2\theta - c^2\cos\theta) \, d\theta$$

$$= \frac{2}{\pi} \left\{ - \frac{3}{8}\phi - \frac{3}{8}\sin\phi \cos\phi - \frac{1}{4}\sin\phi \cos^3\phi + (1+2c)\left(\sin\phi - \frac{\sin^3\phi}{3}\right)\right.$$

$$\left. - (c^2 + 2c)\left(\frac{\phi}{2} + \frac{1}{2}\sin\phi \cos\phi\right) - c^2 \sin\phi\right\}$$

$$= \frac{2}{\pi} \{(\cos^{-1}c)\ (c + \frac{c^2}{2} - \frac{3}{8}) + \sqrt{1-c^2}\ (\frac{7c^3}{4} - \frac{5}{16}c)\}$$

Next

$$\frac{2}{\pi} \int_c^1 (\frac{x^4}{4} + \frac{c^2x^2}{2} - \frac{2cx^3}{3} - \frac{c^4}{12}) \frac{\sqrt{1-c^2}}{\sqrt{1-x^2}} \frac{dx}{x-c}$$

$$= \frac{2}{\pi} \int_c^1 (\frac{1}{12})\ [(\sqrt{1-c^2})\ (3x^3 - 5cx^2 + c^2x + c^3)\ \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{6\pi} \sqrt{1-c^2} \int_0^{\phi} (3\cos^3\theta - 5c\cos^2\theta + c^2\cos\theta + c^3)d\theta$$

$$= \frac{1}{6\pi} \{ c^2\sqrt{1-c^2} - \frac{5}{2} c\sqrt{1-c^2} \cos^{-1}c + 2 + \frac{c^4}{2} - \frac{5c^2}{2} \}$$

Hence

$$g_2(c) = (\cos^{-1}c + \sqrt{1-c^2})\ (-\frac{1}{\pi})\ [(1-2c + 2c^2)\cos^{-1}c - \frac{1}{3}(2c^2 + 9c + 4)\sqrt{1-c^2}\ ]$$

$$- \frac{2}{\pi} [(c + \frac{c^2}{2} - \frac{3}{8}) \cos^{-1}c + (\frac{7c^3}{4} - \frac{5}{16}c)\sqrt{1-c^2}\ ]$$

$$- \frac{1}{6\pi} [c^2 \sqrt{1-c^2} - \frac{5}{2} c \sqrt{1-c^2} \cos^{-1}c + (2 + \frac{c^4}{2} - \frac{5c^2}{2})] .$$

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1. Balakrishnan, A. V.: Active Control of Airfoils in Unsteady Aerodynamics. Appl. Mathematics & Optimization, vol. 4, no. 2, 1978, pp. 171-195.

2. Edwards, John William: Unsteady Aerodynamic Modeling and Active Aeroelastic Control. SUDAAR 504, Stanford Univ., Feb. 1977.

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4. Dowell, Earl H., ed.; Curtiss, Howard C., Jr.; Scanlan, Robert H.; and Sisto, Fernando: A modern course in aeroelasticity. Sijthoff & Noordhoff (Alphen aan den Rijn, The Netherlands), 1978.

5. Sears, William R.: Operational Methods in the Theory of Airfoils in Non-Uniform Motion. J. Franklin Inst., vol. 230, no. 1, July 1940, pp. 95-111.

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T A B L E S

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TABLE 1

Section Parameters

$x_\alpha = 0.2$

$x_\beta = 0.0125$

$r_\alpha^2 = 0.25$ $\qquad \omega_\alpha = 100$ rad/sec

$r_\beta^2 = 0.00625$ $\qquad \omega_\beta = 300$ rad/sec

$a = -0.4$ $\qquad \omega_h = 50$ rad/sec

$c = 0.6$

$b = 1$

$$\mu = \frac{m_s}{\pi\rho b^2} = \frac{m_s}{\pi\rho} = 40$$

$$\frac{\rho}{m_s} = \frac{1}{(40)\pi}$$

$U = 290$

TABLE 2

System matrices: $U = 0;\ \zeta_\beta = 0$

$$M_s = \begin{matrix} 0.100000E+01 & 0.200000E+00 & 0.125000E-01 \\ 0.200000E+00 & 0.250000E+00 & 0.187500E-01 \\ 0.125000E-01 & 0.187500E-01 & 0.625000E-02 \end{matrix}$$

$$K_s = \begin{matrix} 0.250000E+04 & 0.0 & 0.0 \\ 0.0 & 0.250000E+04 & 0.0 \\ 0.0 & 0.0 & 0.562500E+03 \end{matrix}$$

$B_s = 0;\ B_a = 0;\ K_a = 0$

$$M_a = \begin{vmatrix} -0.250000E-01 & -0.999999E-02 & -0.580566E-03 \\ -0.999999E-02 & -0.712499E-02 & -0.473441E-03 \\ -0.580566E-03 & -0.473441E-03 & -0.557130E-04 \end{vmatrix}$$

$$A = \begin{vmatrix} 0.0 & 0.0 & 0.0 & I_{3x3} \\ 0.0 & 0.0 & 0.0 & \\ 0.0 & 0.0 & 0.0 & \\ -0.293400E+04 & 0.251430E+04 & -0.355226E+03 & \\ 0.251430E+04 & -0.147478E+05 & 0.894246E+04 & O_{3x3} \\ -0.157874E+04 & 0.397441E+05 & -0.115730E+06 & \end{vmatrix}$$

TABLE 2: (continued)

$$M_{3,0} = \begin{bmatrix} & 0 & 0 & 0 \\ O_{3x3} & -.0165 & -.01485 & -.00245 \\ & -.0006165 & -.0005549 & -.0000917 \end{bmatrix}$$

$$(I - A_3)^{-1}A = \begin{vmatrix} & O_{3x3} & & I_{3x3} \\ -.002943 & 2465.6 & -451.0 & \\ .002457 & -14636 & 3676.0 & O_{3x3} \\ -.05415 & 45319 & -104830 & \end{vmatrix}$$

TABLE 3

Modes and Eigen Vectors for U = 0

Eigen-values $\pm$ 330.22 j

Eigen-vectors:

1.9×10^{-5}
-2.8×10^{-4}
3.01×10^{-3}
$\pm 6.33 \times 10^{-3} j$
$\mp 9.24 \times 10^{-2} j$
$\pm 9.257 \times 10^{-1} j$

Eigen-values $\pm$ 106.2 j

Eigen-vector

$+ 2.357 \times 10^{-3}$
$- 8.75 \times 10^{-3}$
$- 4.25 \times 10^{-3}$
$\pm 2.50 \times 10^{-1} j$
$\mp 9.297 \times 10^{-1} j$
$\mp 4.51 \times 10^{-1} j$

Eigen-values $\pm$ 47.985 j

Eigen-vectors

2.00×10^{-2}
5.64×10^{-3}
2.39×10^{-3}
$\mp 9.61 \times 10^{-1} j$
$\mp 2.71 \times 10^{-1} j$
$\mp 1.15 \times 10^{-1} j$

F I G U R E S

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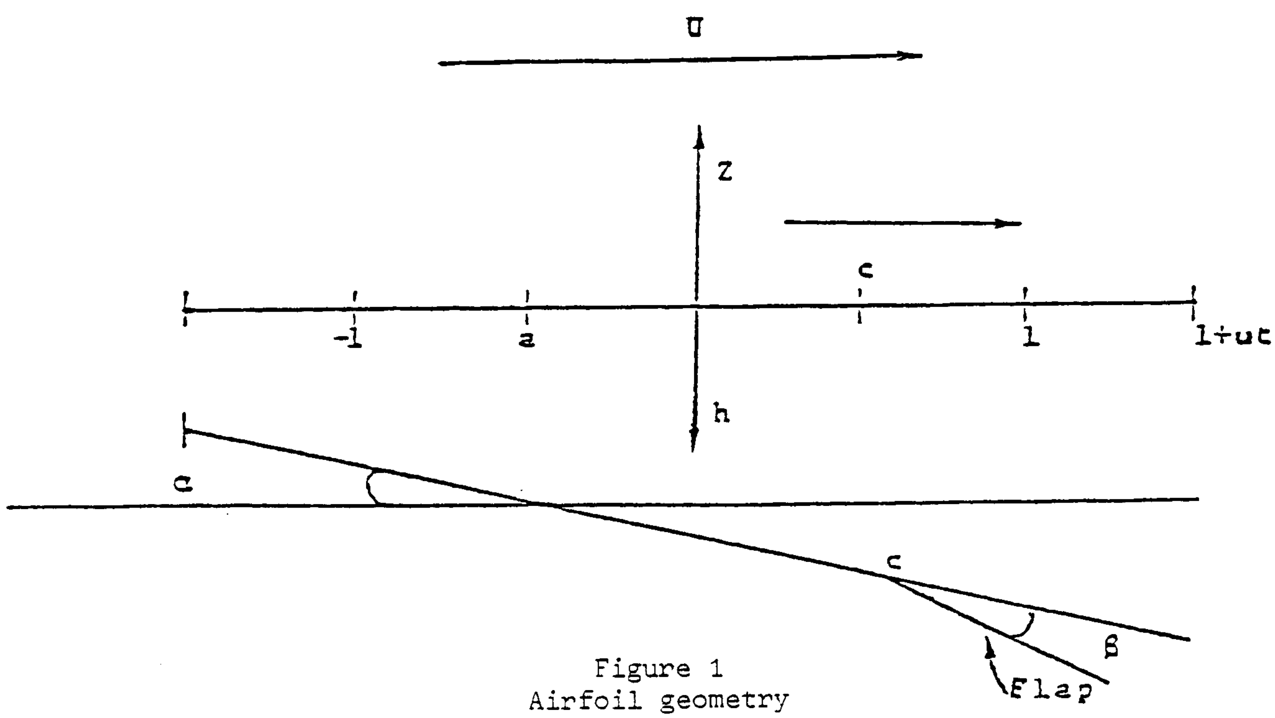


Figure 1
Airfoil geometry

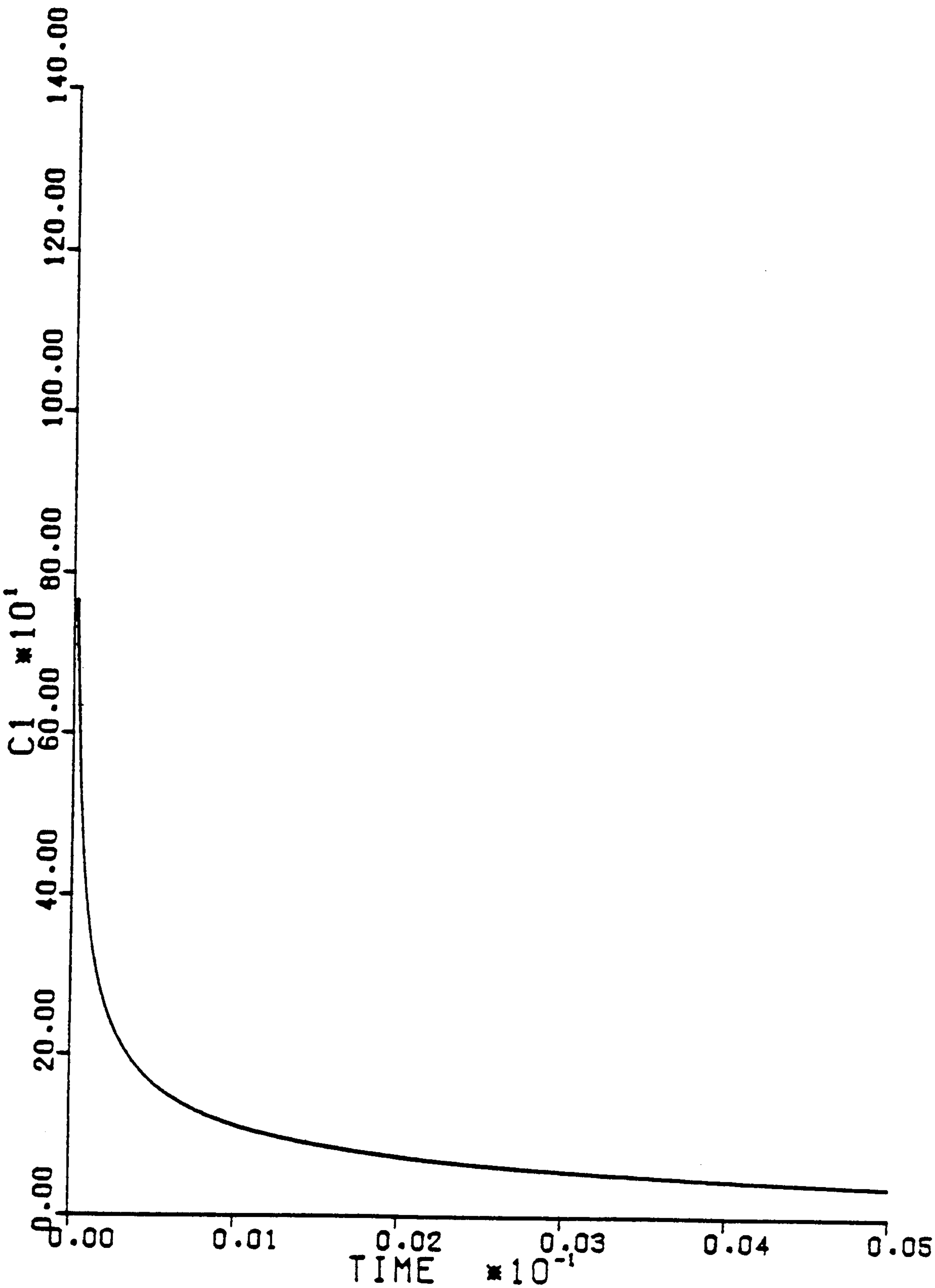


Figure 2

Function c_1 (t)

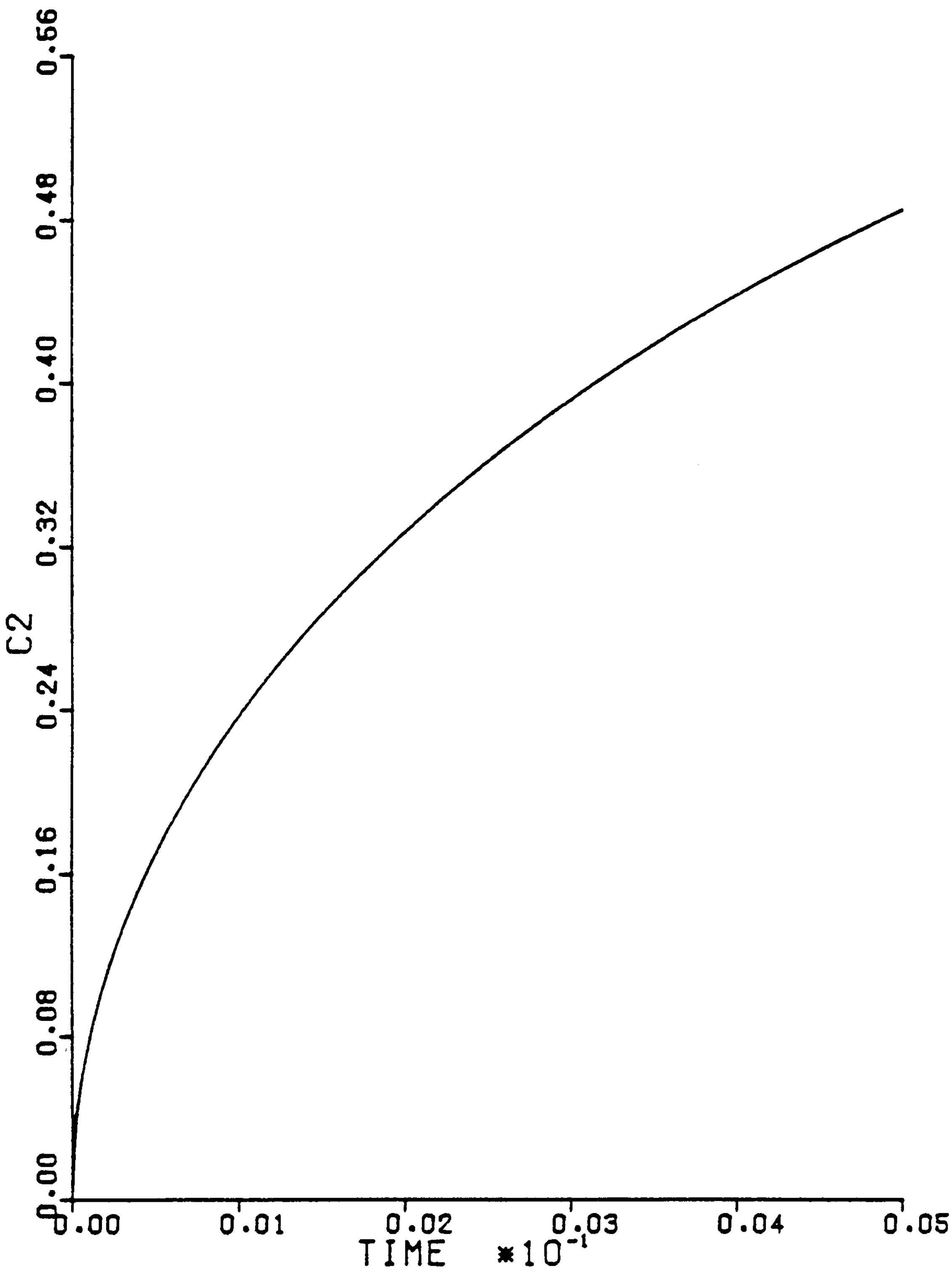


Figure 3

Function c_2 (t)

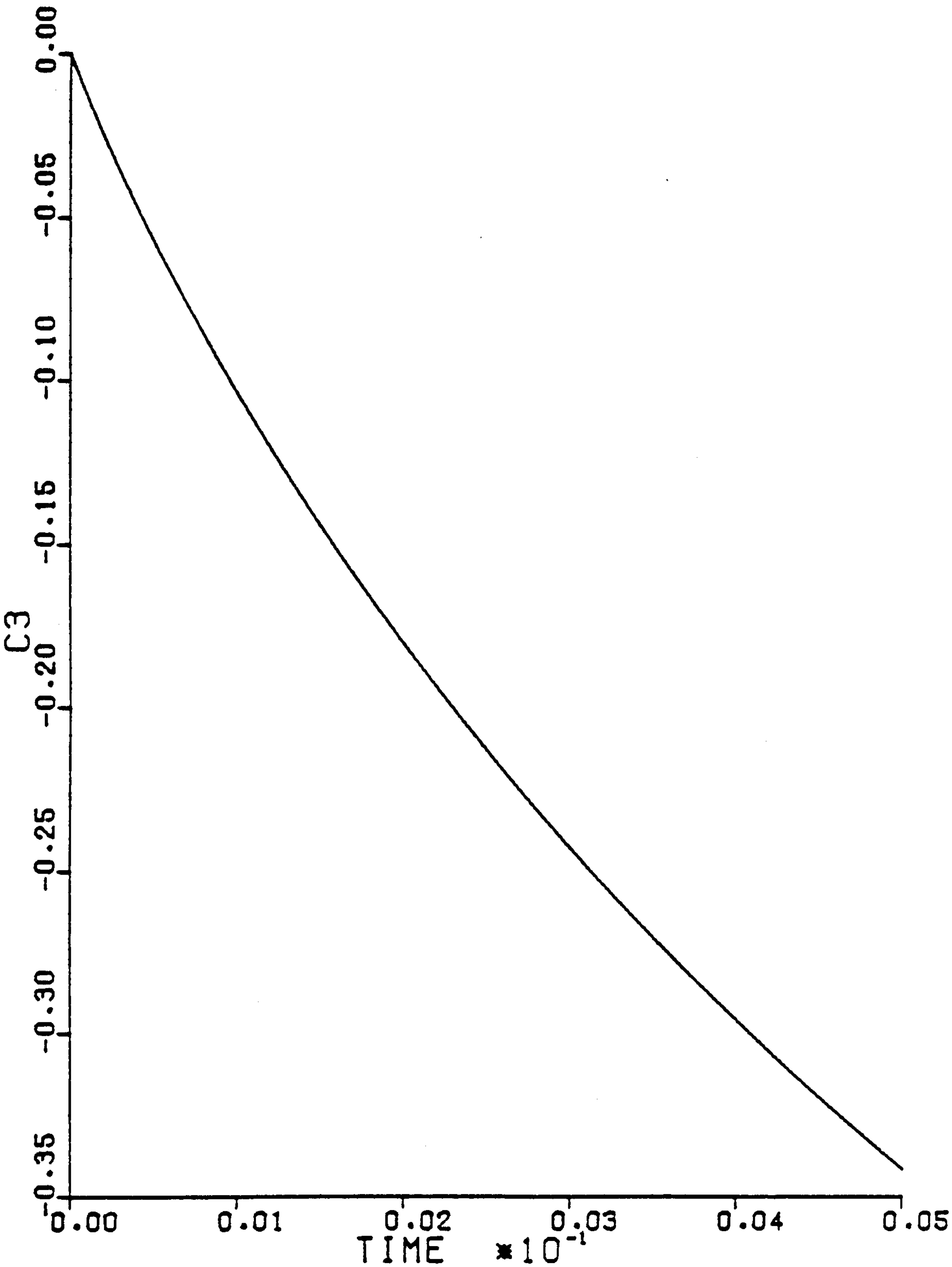


Figure 4

Function c_3 (t)

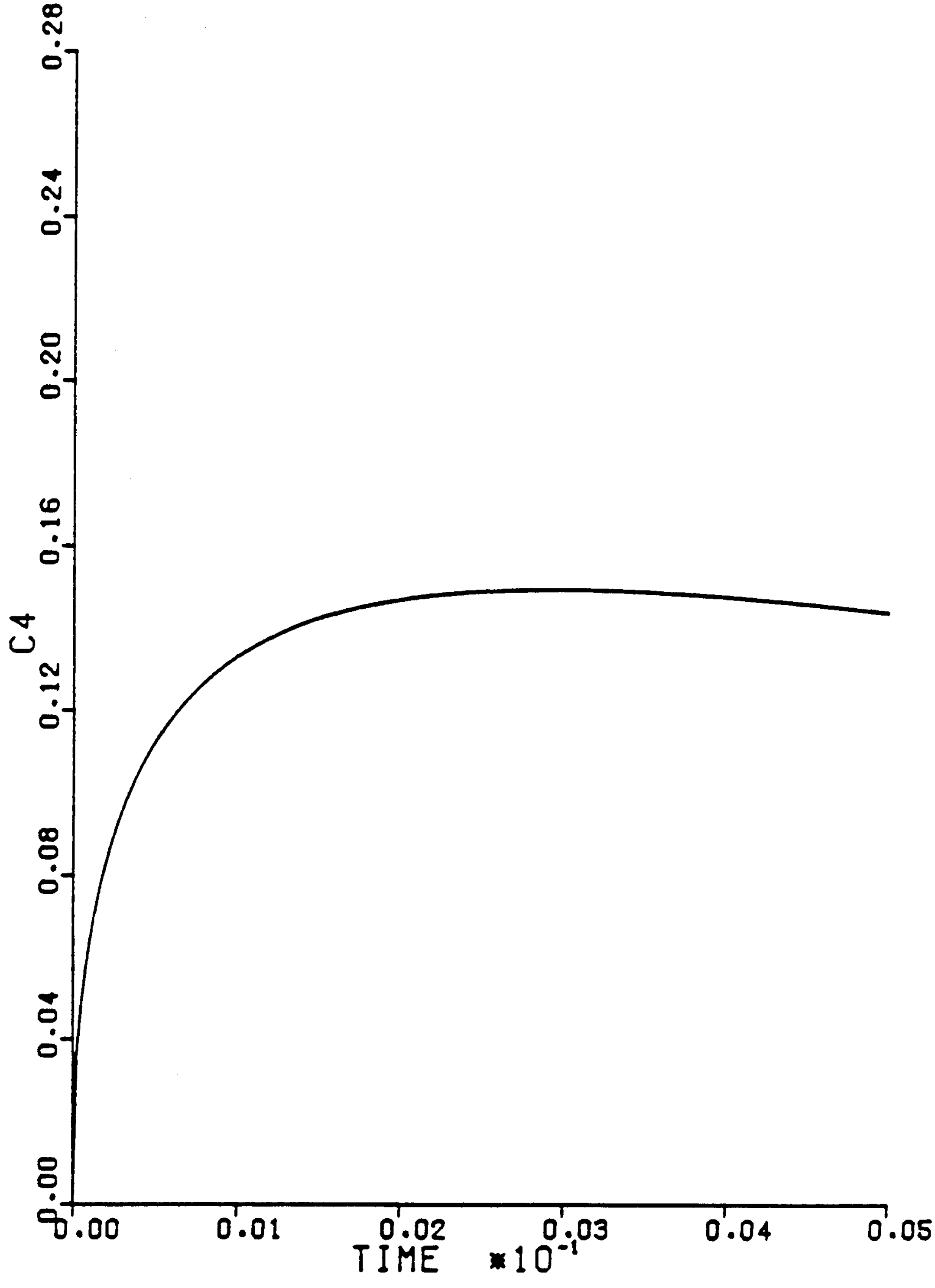


Figure 5

Function c_4 (t)

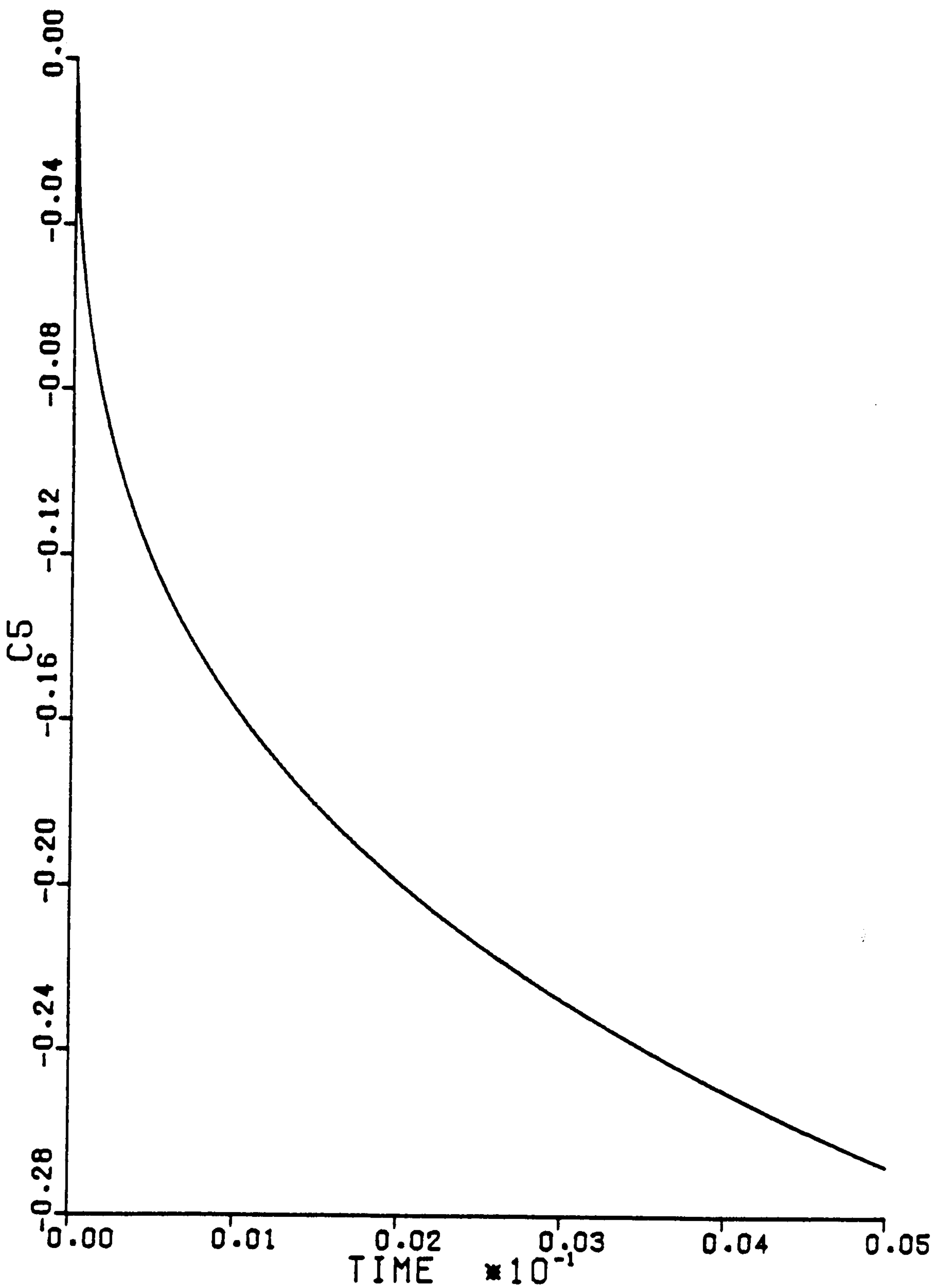


Figure 6

Function c_5 (t)

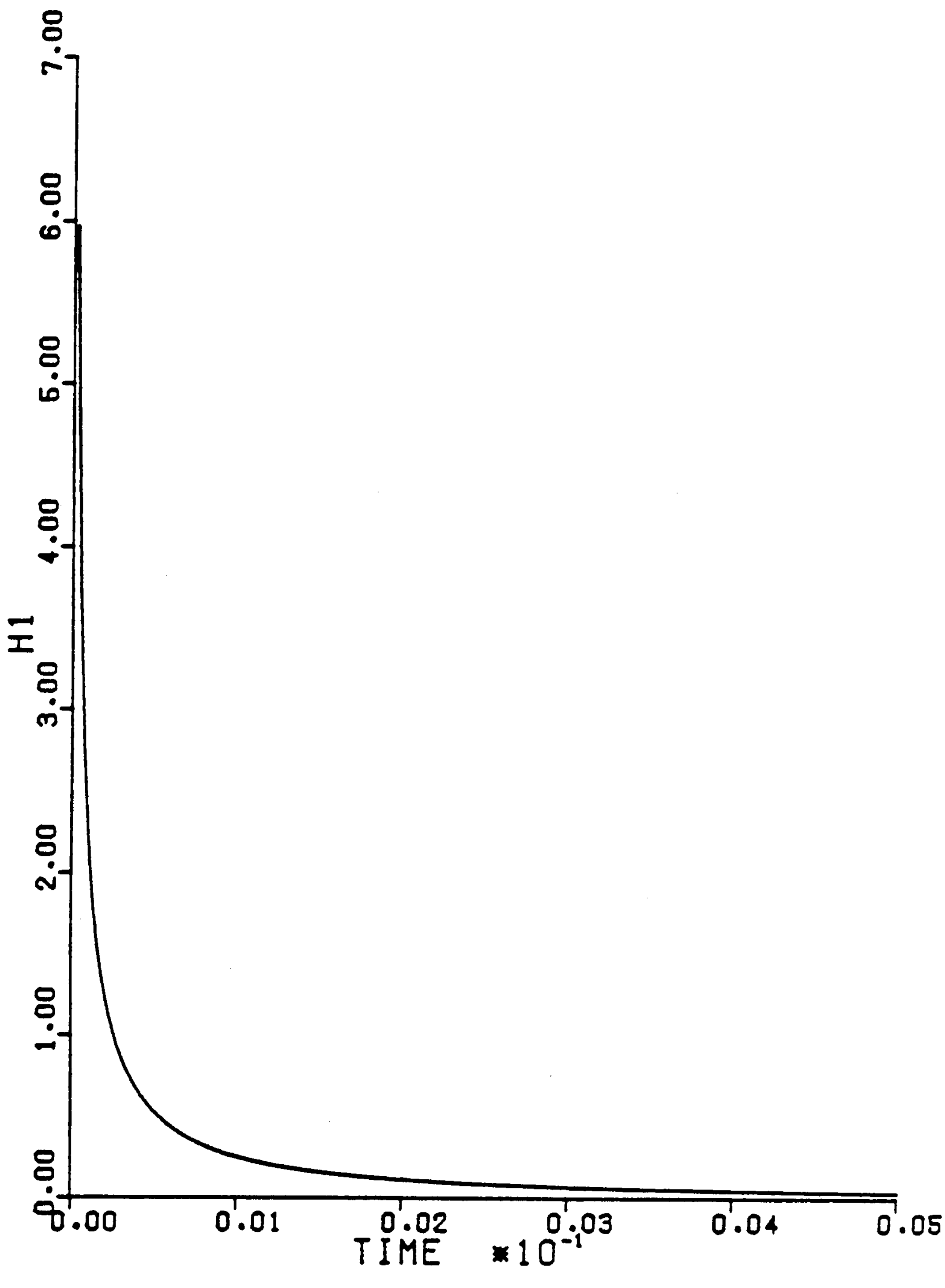


Figure 7

Function H_1 (t)

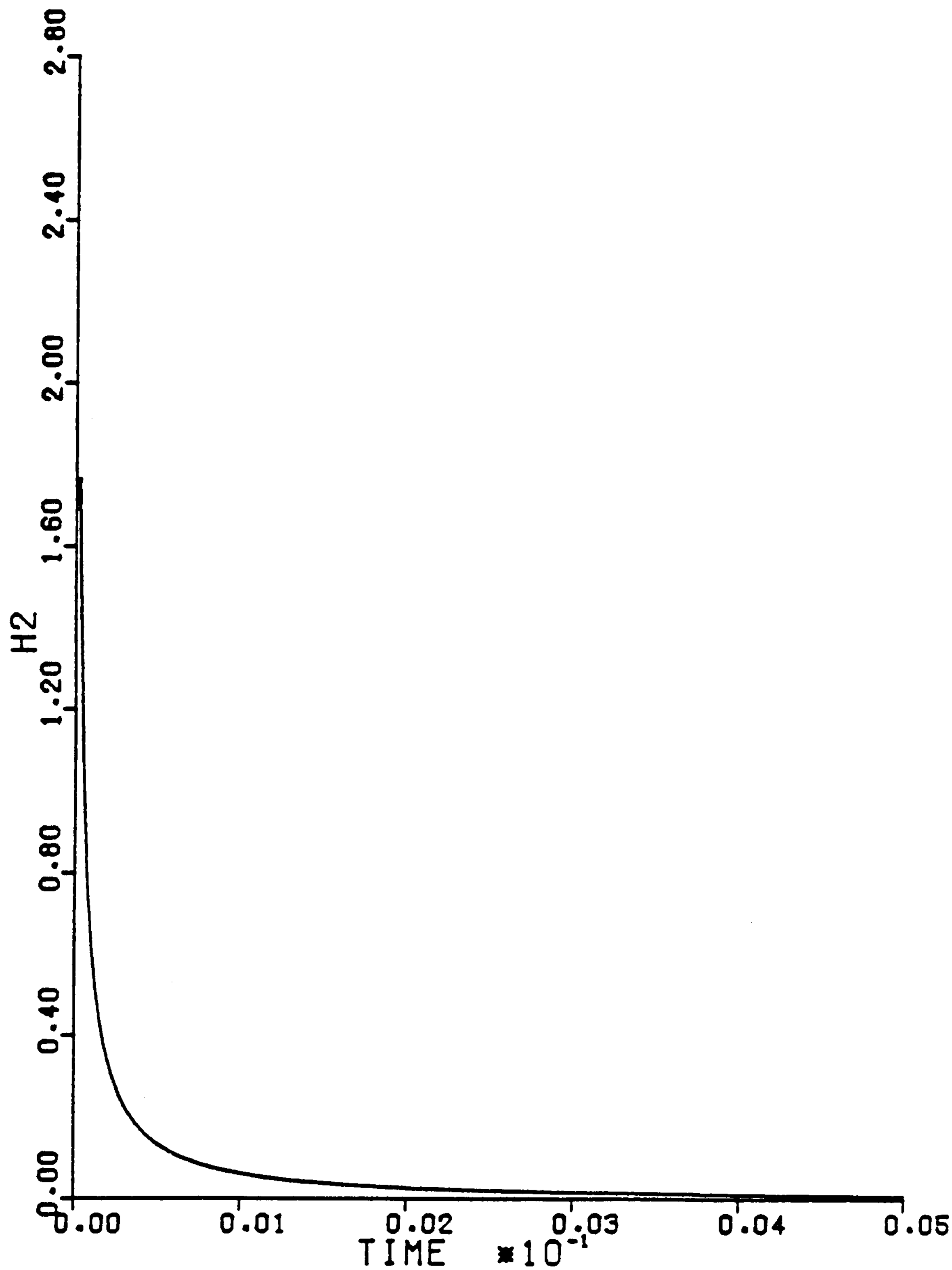


Figure 8

Function H_2 (t)

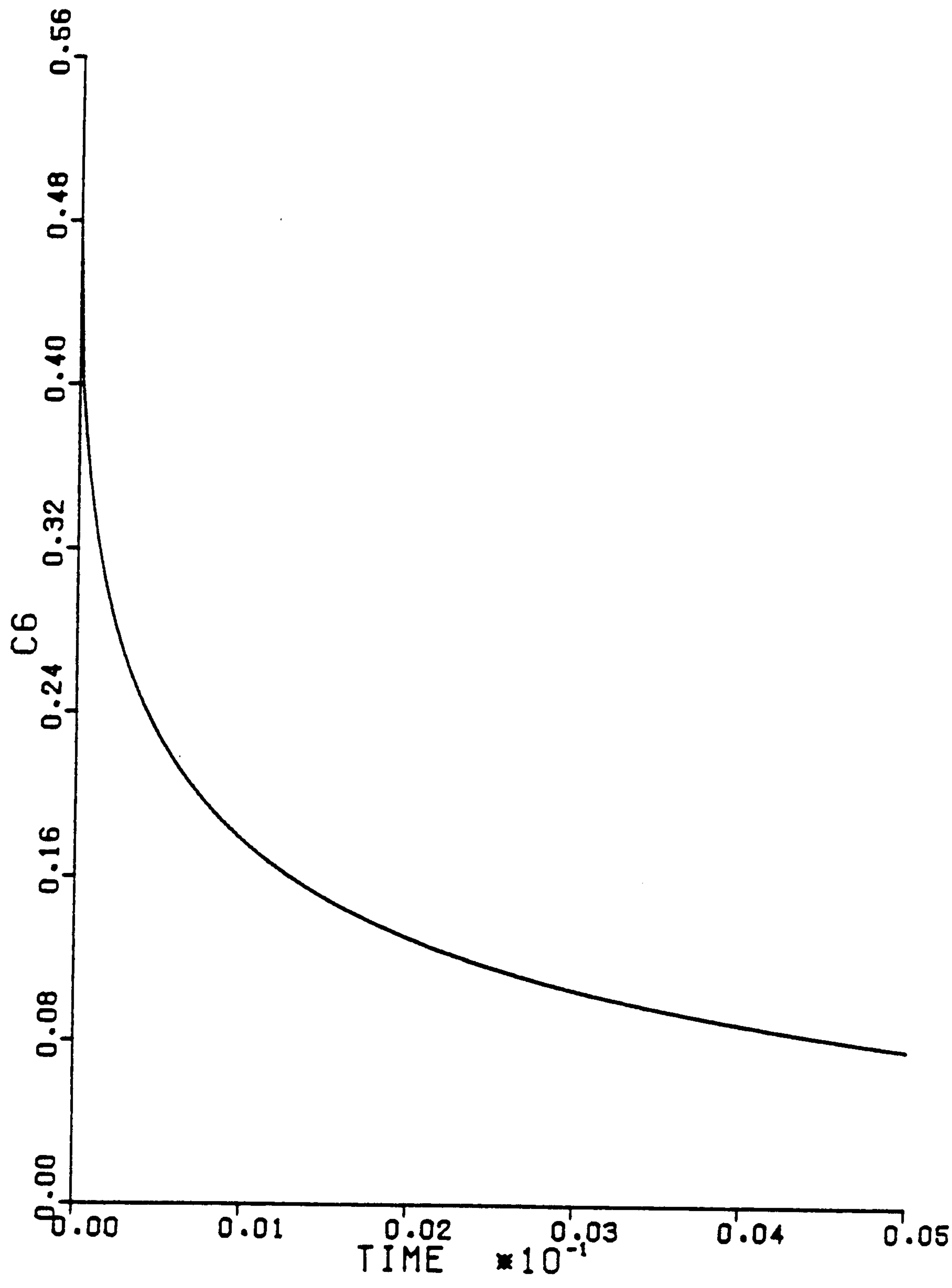


Figure 9

Function c_6 (t)

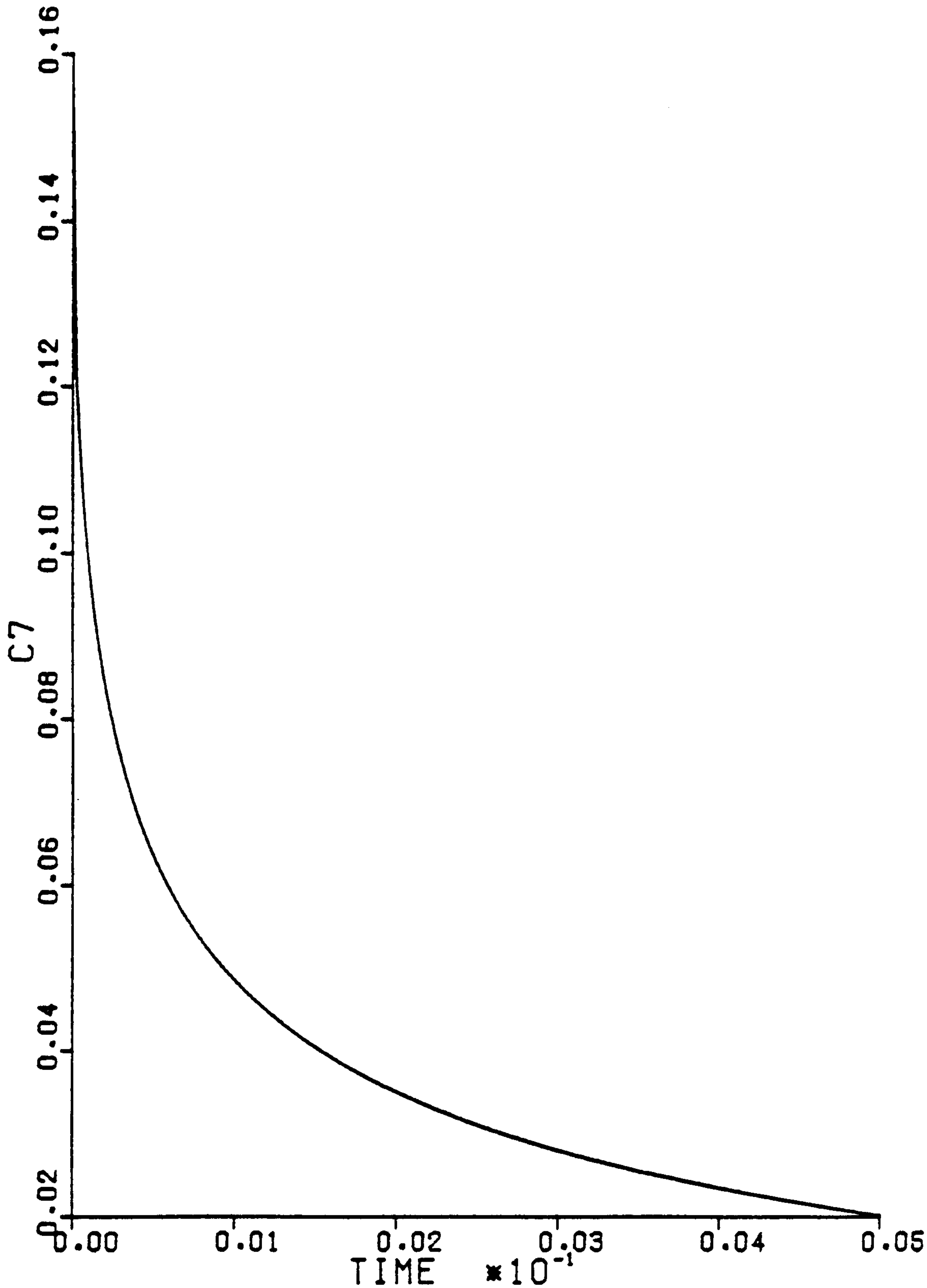


Figure 10
Function c_7 (t)

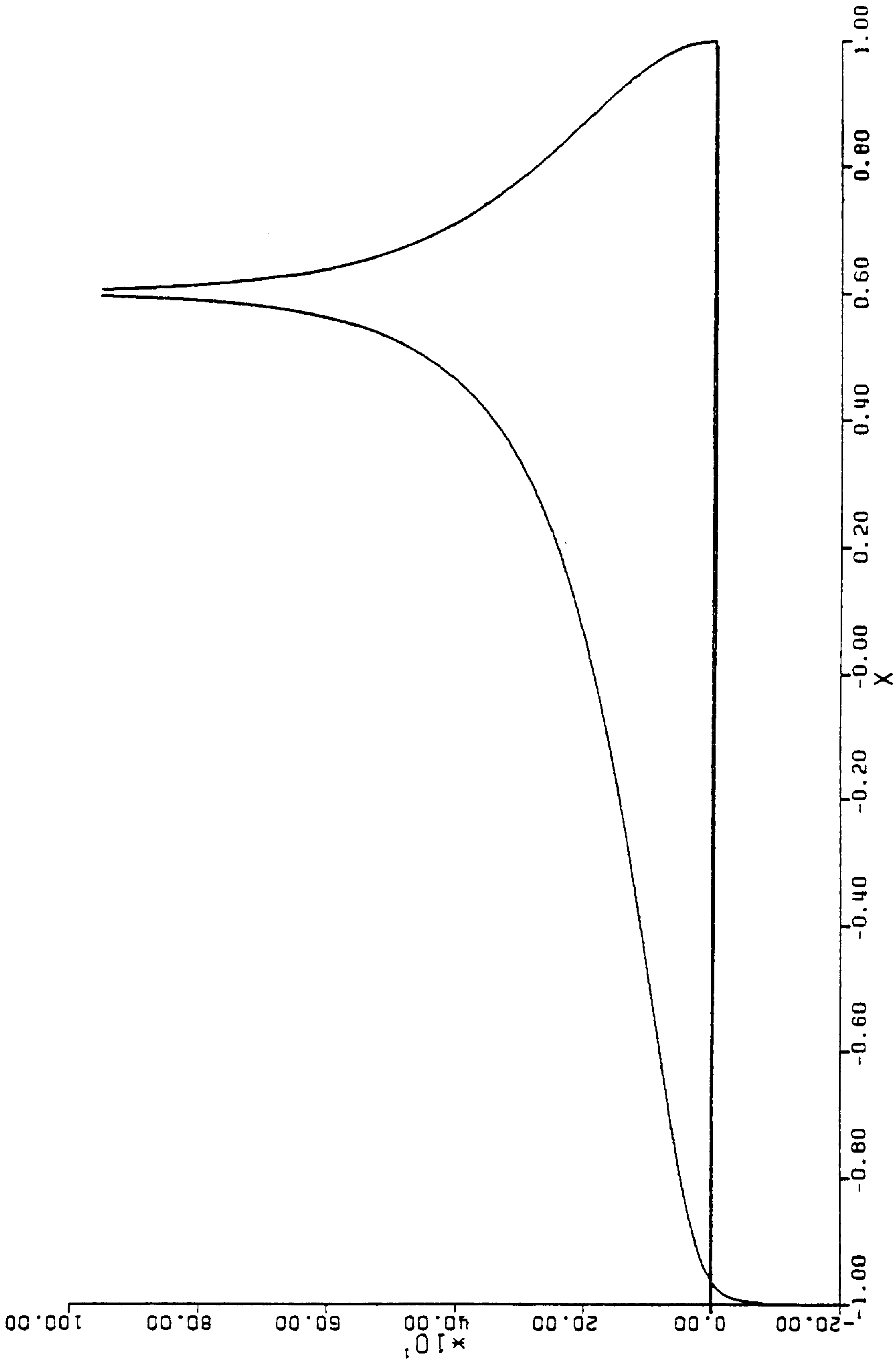
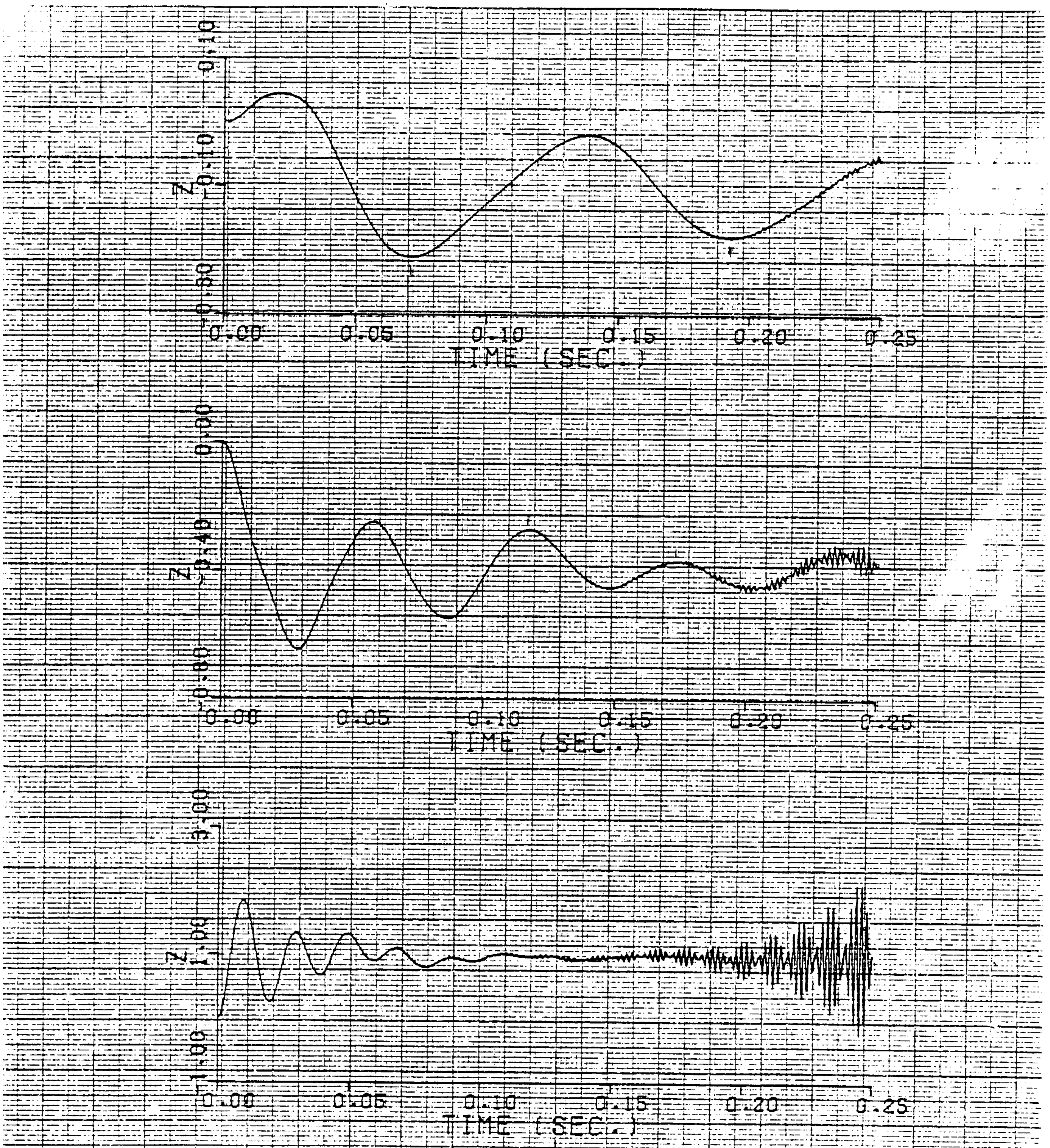


Figure 11

Steady State Circulation on Foil due to Flap Deflection

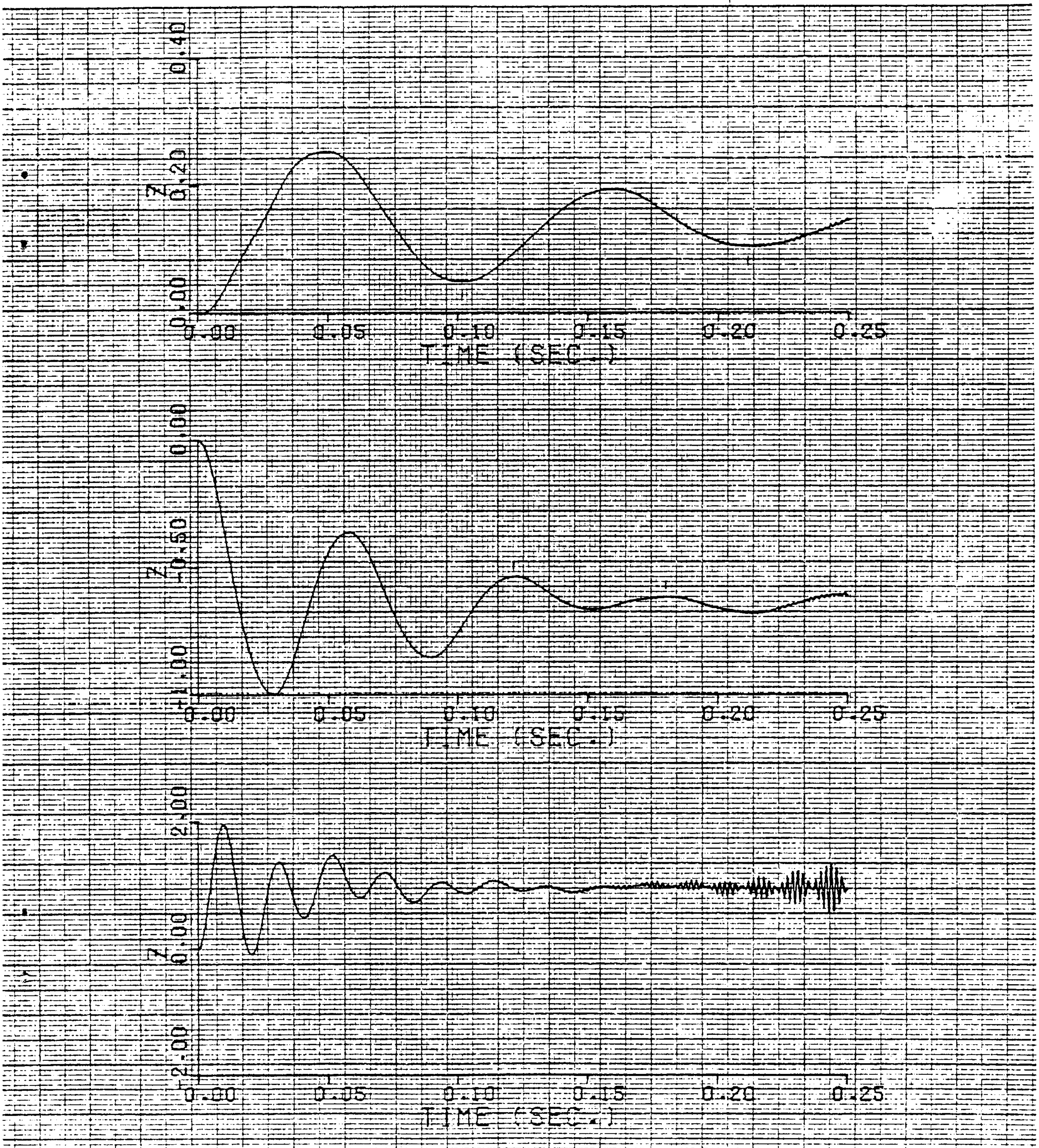
Steady State Circulation on Foil due to flap deflection



Transient Response

Fig. 12a

U = 250.00 ZETAB = 0.1000



Transient Response

Fig. 12b

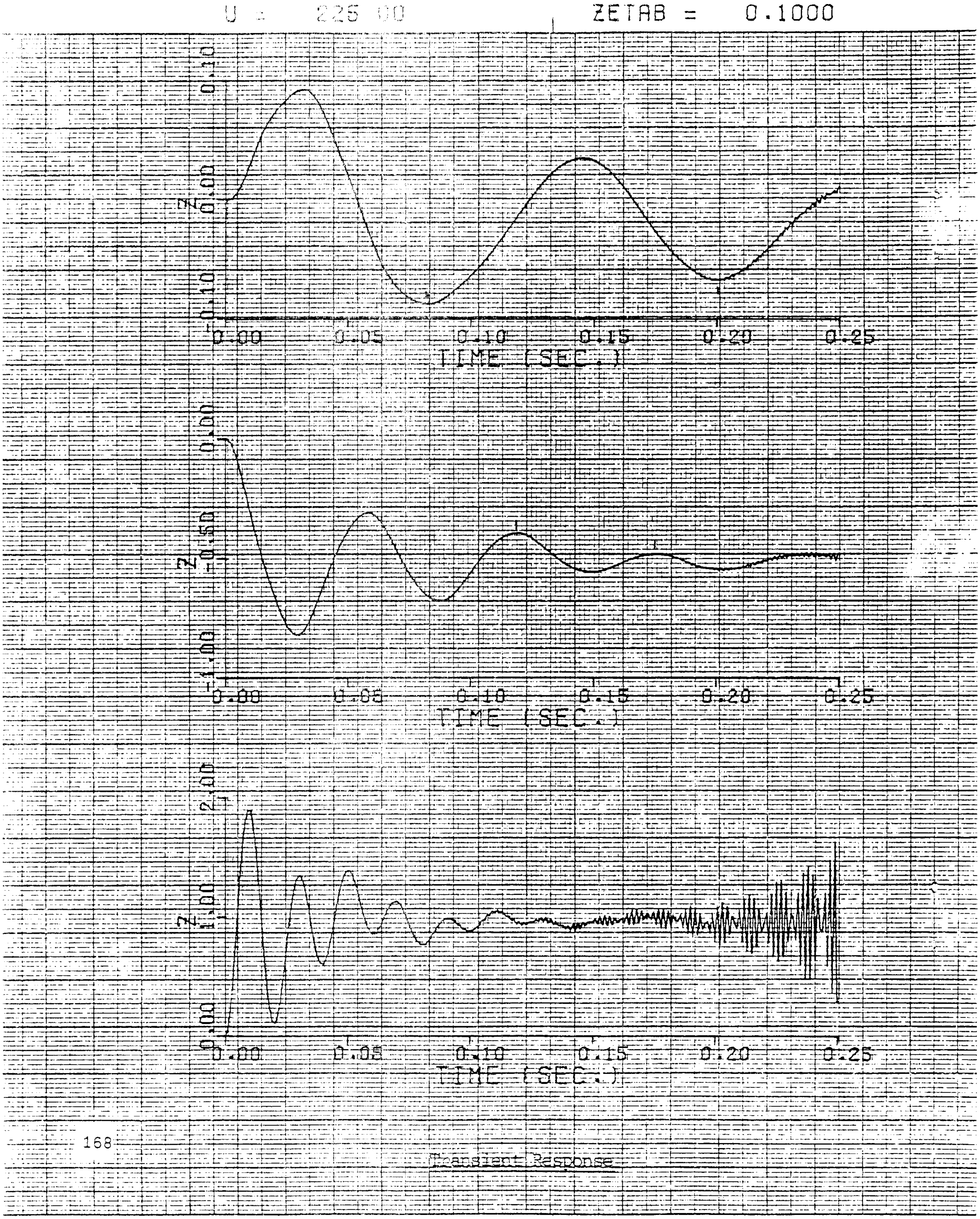
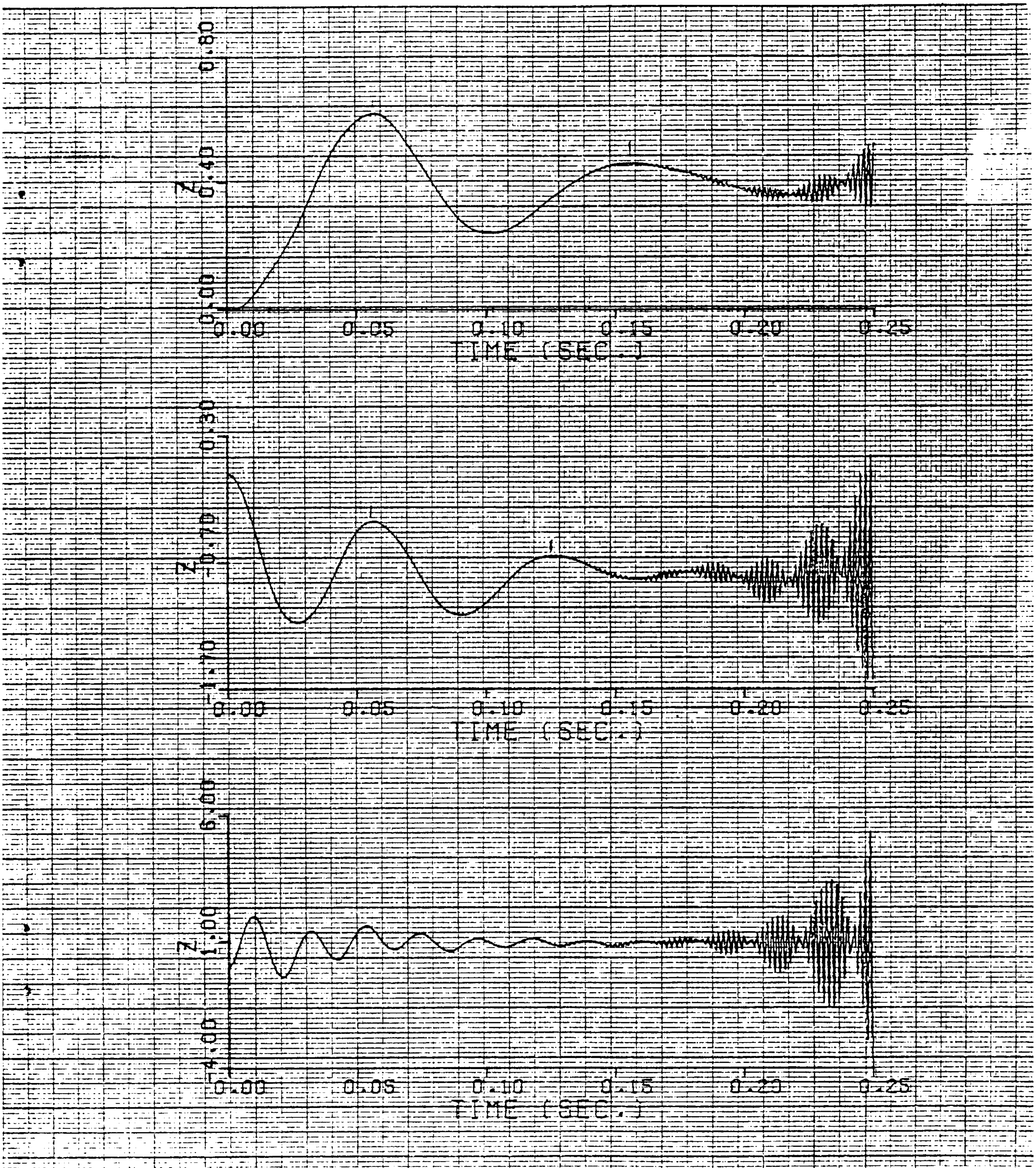

U = 225.00
ZETAB = 0.1000
0.10
0.00
-0.10
Z
0.00
0.05
0.10
0.15
0.20
0.25
TIME (SEC.)
0.00
-0.50
-1.00
Z
0.00
0.05
0.10
0.15
0.20
0.25
TIME (SEC.)
2.00
1.00
0.00
Z
0.00
0.05
0.10
0.15
0.20
0.25
TIME (SEC.)
Transient Response

Fig. 12c

U = 275.00 ZETAB = 0.1000



Transient Response

Fig. 12d

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16. Abstract

The time-domain equations of motion of elastic airfoil sections forced by control surface motions and gusts are developed for the case of incompressible flow. Extensive use is made of special functions related to the inverse transform of Theodorsen's function. Approximations are given for the special cases of zero stream velocity, small time, and large time. A numerical solution technique is given for the solution of the general case and examples of the exact transient response of an airfoil are presented.

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